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Totter

A NEW INTRODUCTION TO THE SCIENCE OF ALGEBRA;

DESIGNED FOR STUDENTS IN COLLEGES, AND THE HIGHER SCHOOLS
AND ACADEMIES.

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PREFACE.

WHILE engaged as a teacher of elementary mathematics, the author has often felt the want of a Treatise on Algebra, which might serve as an introduction to the higher, and more difficult parts of that science; a treatise which, commencing with the first principles of Arithmetic, might conduct the learner by gradual and easy steps, from the expression of quantity by numbers, to the investigation of the relations of quantity, by algebraic symbols. In most of our public schools, the pupil commences the study of Algebra, with a very imperfect knowledge of arithmetic. He has been taught arithmetic, it is true, but *mercantile* arithmetic only. He has been taught to perform operations upon numbers, by rules, the principles of which, he is rarely required to investigate, and which, for the most part, the books that are put into his hands, do not even profess to demonstrate. He thus begins the study of algebra, unacquainted with the first principles of mathematical science, and finds himself perplexed at the same time with a kind of reasoning, to which he is wholly unaccustomed, and new symbols for the expression of quantity, the nature and use of which, he finds it difficult to comprehend. With these difficulties to encounter, it is not strange that all seems dark and confused; or, that he often turns, with disappointment and disgust, from a study, upon which he was ill prepared to enter.

To remove these difficulties, is the object of the short treatise on arithmetic, with which this volume commences. In it, an attempt has been made to explain, with clearness and brevity, the connexion between numbers, and the quantities they are used to express; and also to give concise and satisfactory demonstrations of the rules for numerical calculations. The second chapter is necessarily more full than any other part of the arithmetic, inasmuch as a thorough knowledge of fractions, both vulgar and decimal, is an indispensable preparation for the study of algebra, and the same demonstrations apply both to numerical and algebraic fractions.

Having accustomed the pupil to reasoning upon numbers, in the first three chapters, letters, as the representatives of numbers, are first introduced in the fourth; and the same reasoning applied to them, which immediately before, had been applied to the numbers themselves, thus preparing the way for the use of algebraic symbols, and making the transition from arithmetic to algebra easy and pleasant.

The treatise on algebra, which forms the principal part of the present volume, is not so closely connected with the arithmetic as to render it necessary that both should be studied together. If the pupil be well versed in the principles of arithmetic, he may proceed at once with algebra; yet he will find it convenient to have always at hand, a treatise on arithmetic to which he can refer, when he would recall some principle which he may have forgotten.

It was the author's intention, that the treatise on algebra should be eminently practical, and for this purpose a great number of examples have been introduced. The examples are so selected, as to illustrate, most fully, the rules to which they are applied, and so arranged as to form a series of progressive exercises, rendering the passages from the more simple, to the most complicated algebraic operations, as easy as possible, and preparing the student, by making him familiar with the practical part of algebra, to enter with advantage upon the higher branches of mathematical science. Throughout the whole, the reasoning is adapted to the progress of the student. In the first part of the work, the rules are given and applied, and then followed by demonstrations; but as the student is supposed to become more familiar with mathematical reasoning, the analytic method is gradually introduced; and in the last two chapters, used to the exclusion of every other. This is an important part of the plan of the work, it being originally designed to introduce the student, by a series of progressive exercises, to the more abstruse analytic reasoning of the higher branches of algebra. It is hoped, however, that while it meets the wants of students who are commencing an extensive course of mathematics, it will be particularly useful to those who have less leisure for study, or who, on account of their particular profession, or occupation in life, are desirous only of obtaining a knowledge of the more practical and useful parts of mathematical science.

WASHINGTON COLLEGE, *Hartford, Aug.* 1836.

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INTRODUCTION TO ALGEBRA.

CHAPTER I.

PRELIMINARY REMARKS AND DEFINITIONS.

(1.) Quantity, or magnitude, is any thing whatever that will admit of increase or diminution; for example, a sum of money, the length of a line, and a given weight, or bulk, are quantities, since they can be increased or diminished. Mathematics is the science that investigates the means of measuring quantity, and hence is called the *science of quantity.*

A quantity cannot be great or small, much or little in itself; it is only by comparing it with another quantity of the same kind that it becomes so. Thus, a mile is a large quantity compared with an inch, but small compared with the diameter of the earth; and this again is a small quantity compared with the distance of the earth from the sun.

(2.) It is evident, therefore, that in order to measure a quantity, we must compare it with some known quantity of the same kind, assumed as the unit or measure of quantity. For example, if the quantity to be measured is a sum of money, and we assume one dollar as the unit, the number of times one dollar is contained in that quantity, is the magnitude of the quantity. Thus 300 dollars is conceived to be composed of 300 parts, each of which is equal in value to one dollar.

If the quantity be the length of a line, and we assume one foot as the unit of measure, the number of times a line one foot in length can be applied to this line, is its magnitude. If it can be applied 10 times, we say the given line is ten feet in length.

Quantities of every kind whatever can therefore be expressed by numbers, the number always expressing how many times the assumed unit of quantity is contained in the given quantity.

(3.) Arithmetic and Algebra both treat of the relations of quantity as expressed by numbers.

Arithmetic treats of numbers in particular, and their simplest combinations. Algebra is but a continuation of arithmetic, in which proper signs are used, to abridge and generalize the reasoning required in the resolution of questions relating to numbers.

ARITHMETIC.

NOTATION AND NUMERATION.

(4.) It would evidently be impossible to express each number by a separate term, since in that case the number of terms would be infinite. To express the numbers comprehended between one and one million, a million of terms would be necessary; and as any number, however large, may be increased without limit, by adding unity to it successively, the number of terms would also increase without limit.

To obviate this difficulty, a nomenclature has been devised in which, by the use of a small number of terms properly combined, and subject to regular forms, all possible numbers can be expressed.

This nomenclature is based upon the principle of successive orders of units, of such values that ten units of the first order shall be equal to one of the second order, ten of the second order equal to one of the third order, and so on. We number to ten, and then commence again at one; eleven, twelve, thirteen, &c., being the same as ten and one, ten and two, ten and three, &c., till we have numbered another ten, when we commence at unity as before: twenty-one, twenty-two, &c. Proceeding in this manner till we have numbered ten tens, we pass to hundreds, and in like manner from hundreds to thousands.

(5.) Numbers were at first written separately, either in words at length, or as among the Romans, in characters, commonly the letters of the alphabet; and for large numbers, such letters were used as would make up, by the addition of their separate values, the number required. Thus the number four hundred and fifty-eight was written CCCCLVIII., the value of C being one hundred, L fifty, V five, and I one.

The system of notation now in use, and which originated in Arabia, is so contrived as to express all numbers with ten characters only. Nine of these are significant, and represent numbers, and the other is used to denote *nothing*, or the absence of quantity.

To express numbers greater than nine, recourse is had to a law which assigns different values to the figures, according to the position which they occupy. According to this law, units of the first order occupy the first place on the right of the written expression, units of the second order, the second place, and so of the others.

The numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, are each expressed by separate characters ; but when we would write ten, which is a unit of the second order, we place unity in the second place, and a 0 in the first, thus, 10, which may either be considered as one unit of the second order, or ten units of the first. Proceeding with units of the second order in the same manner as with those of the first, we shall have, 10, 20, 30, 40, 50, 60, 70, 80, 90, and next a unit of the third order, which is written 100, the significant figure being written in the third place. In like manner we pass from the third to the fourth order.

A number may consist of units of more than one order, and then the places of the 0's are supplied by significant figures. For example, the number three hundred and fifty-six has six units of the first order, five of the second, and three of the third ; and writing these units in their proper places, we have the expression 356.

The following table can now be understood, which exhibits the successive orders of units, and the manner of writing and reading numbers.

5th Period.	Hundreds of trillions.	4	15th.
	Tens of trillions.	1	14th.
	Trillions.	2	13th.
4th Period.	Hundreds of billions.	3	12th.
	Tens of billions.	7	11th.
	Billions.	4	10th.
3d Period.	Hundreds of millions.	9	9th.
	Tens of millions.	6	8th.
	Millions.	1	7th.
2d Period.	Hundreds of thousands.	2	6th.
	Tens of thousands.	4	5th.
	Thousands.	3	4th.
1st Period.	Hundreds.	9	3d.
	Tens.	8	2d.
	Units.	7	1st order.

This table contains fifteen places of figures, and, consequently, fifteen orders of units, and the number expressed is read thus: four hundred and twelve trillions, three hundred and seventy-four billions, nine hundred and sixty-one millions, two hundred and forty-three thousand, nine hundred and eighty-seven.

It will be observed in reading the number written in the table, that for each period of three figures, the same names are used as for the first at the right, with an additional term to express a higher order of units. This, in the second period, is thousands; in the third, millions; in the fourth, billions; and in the fifth, trillions; and hence in writing numbers, it is usual to divide the figures into periods of three each, as follows:

412,374,961,243,987.

To express a number which does not contain units of every order comprehended between the lowest order and the highest contained in the given number, the places of such as are wanting are to be filled up with 0's; for example:

Two hundred trillions, twenty billions one million, twenty-two thousand and seven, is written

200,020,001,022,007.

EXAMPLES FOR EXERCISE.

1. Write thirty thousand one hundred and six.
2. Write one million one thousand one hundred and one.
3. Write thirty billions, two millions, one hundred and one thousand and twenty-seven.
4. Write sixteen billions, six millions one thousand and seven.
5. Write eleven thousand eleven hundred and eleven.

EXPLANATION OF SIGNS.

To denote arithmetical operations and conditions, certain characters or signs are used; they are the following:

+, which is read *plus*, and indicates Addition.
—, which is read *minus*, and indicates Subtraction.
×, which indicates Multiplication.
÷, " " Division.
: = : " " Proportion.

$=$, which signifies Equality.
$\sqrt{\ }$ " " Square root.
$\sqrt[3]{\ }$ " " Cube root.
$\sqrt[4]{\ }$ $\sqrt[5]{\ }$ $\sqrt[6]{\ }$, &c. which signifies 4th, 5th, 6th root, &c. Thus:
$6 + 8$ denotes that 6 is to be added to 8.
$6 - 4$ denotes that 4 is to be subtracted from 6.
7×3 or 7.3 denotes that 7 is to be multiplied by 3.
$16 \div 4$ or $\frac{16}{4}$ denotes that 16 is to be divided by 4.
$9 : 6 = 12 : 8$, shows that 9 is to 6 as 12 is to 8.
$8 + 4 = 12$ shows that the sum of 8 and 4 is equal to 12.
$\sqrt{6}$ signifies the square root of the number 6.
$\sqrt[3]{6}$ signifies the cube root of the number 6.
$\sqrt[4]{2}$, $\sqrt[5]{2}$, $\sqrt[6]{2}$, &c. signifies the 4th, 5th, 6th, &c. root of the number 2.

THE FOUR RULES OF ARITHMETIC.

(6.) As quantity admits of no other changes than increase and diminution, it is evident that all the operations of arithmetic must be based upon these only.

Under the first are comprehended addition and multiplication, and under the second, subtraction and division.

Addition consists in finding the sum of two or more numbers.

Multiplication is the successive addition of a number to itself a given number of times.

Subtraction consists in finding the difference between two numbers, or diminishing one given number by another.

Division is the same as subtracting one number successively from another, in order to find how many times the smaller number is contained in the larger.

Addition, subtraction, multiplication, and division, are called the four rules of arithmetic; all arithmetical operations whatever are combinations of these.

ADDITION.

(7.) Addition consists in finding the sum of two or more numbers. The sign of addition is +, and is read *plus*; $5 + 6$ signifies that 6 is added to 5.

RULE.

Set down the numbers so that the units of each order in the several numbers, may stand in the same vertical column, units under units, tens under tens, hundreds under hundreds, &c. Then find the sum of the figures in the right hand column, and see how many tens are contained in that sum; set down the excess above the tens, and carry one for every ten to the next column. Proceed in the same manner with all the other columns to the last, and under it set down the whole sum.

Demonstration of the Rule.—This rule depends upon the principles of notation, (5) for since ten units of the first order are equal to one of the second order, when we have taken the sum of the units of one order, and find it more than ten, we have units of a higher order, which cannot be expressed in our system of notation, except by putting them in the next place to the left, which is evidently the same as adding them in the sum of the next column. The annexed example illustrates this principle.

$$\begin{array}{r} 41642\\ 53841\\ 37159\\ \hline 12\\ 13\\ 15\\ 11\\ 12\\ \hline 132642 \end{array}$$

In this example, the sum of the units is 12, which cannot be otherwise expressed, than by putting the figure 1 in the place of tens. The sum of the tens is 13, which is written by putting 1 in the place of hundreds, and so of the others. Now, by taking the sum of all these partial sums, we shall evidently have the whole sum; but this does not differ in its result from the method pointed out in the rule. In the one case, the tens are written down in their places, and then added together; and in the other they are retained in the memory to be added to the next column.

EXAMPLES.

23141 + 31811 + 98365 + 372 + 19 =
71134627 + 351 + 2 + 517356 =
1262 + 3557 + 4634 + 12 + 7376 + 93172 =
71347 + 1 + 63 + 71212 + 93672934 =

Proof of Addition.—Addition may be proved by beginning at the top of the column, and adding the figures in a contrary direction, or by cutting off by a line the upper number, and taking the sum of all the numbers below it, and then adding this sum to the upper number. If, in either case, the result agrees with the former, the work is presumed to be correct.

SUBTRACTION.

(8.) Subtraction consists in finding the difference between two numbers. The greater of the two numbers is called the *minuend*, the less the *subtrahend*, and the difference obtained, the *remainder*. The sign of subtraction is —, and is read *minus*; 8 — 4 signifies, that 4 is to be subtracted from 8.

RULE.

Write the numbers, as in addition, placing the less number under the greater. Begin at the right, and take each figure of the subtrahend from the corresponding one of the minuend; but if the figure in the minuend be the least, add ten to it, and subtract as before, always in such cases carrying one to the next figure of the subtrahend, to compensate for the ten added to the minuend.

Demonstration of the Rule.—To show the correctness of this rule, we remark that the difference of two numbers is evidently the difference of the units, tens, hundreds, &c. which compose these numbers; and hence, when the figures in the subtrahend are all less than those in the minuend, the sum of the differences of the several orders of units will be the difference of the numbers. When the figure in the minuend is less than the corresponding one in the subtrahend, and the rule requires us to add ten to it;—by doing this we but borrow one from the next higher order, which becomes ten by moving it one place to the right; the figure from which this was borrowed ought therefore to be diminished by one, or the figure below it increased by one, which evidently gives a like result.

Proof.—The proof of subtraction consists in adding together the remainder and the subtrahend; the sum must equal the minuend.

Examples

1. From 9092 subtract 7835

$$\begin{array}{r}9092\\7835\\\hline 1257\end{array}$$

Having written the numbers as directed in the rule, I commence at the right, and since 5 cannot be taken from 2, I borrow 1 from the next place to the left. This, moved one place towards the right, becomes 10, which added to 2, gives 12, from which 5 can be subtracted. The remainder 7 is written below, and I proceed to the next place of figures to the left. Now since 1 was borrowed from 9, I may either consider it diminished by one, and take 3 from 8, or adding 1 to 3 take 4 from 9; either of which methods gives 5 for a remainder. Proceeding to the next figure, since 8 cannot be subtracted from 0, I add 10 to it: taking 8 from 10, 2 remains, and 1 is to be carried to 7, which gives 8 to be subtracted from 9, the remaining figure of the minuend. Writing the difference 1, below, I find the whole difference 1257, which added to the subtrahend, will produce the minuend.

If the left hand figure in the minuend had been less than the figure immediately below it in the subtrahend, the subtraction would have been impossible, since there are no figures to the left from which to borrow. Indeed, in that case, the minuend would have been less than the subtrahend; and it is evident that the greater number cannot be taken from the less.

2. 322974 — 123748 = 199226
3. 1002641 — 783219 =
4. 10000032 — 1364217 =
5. 666444 — 55455 =

6. From 3 millions 31 thousands and 1, take 3721.

7. From 1 billion 100 millions 2 thousands and 2, take 810,032.

8. From 1 thousand 1 hundred and 11, take 112.

MULTIPLICATION.

(9.) Multiplication is the successive addition of a number to itself a given number of times. The sign of multiplication is ×; 8×3 signifies that 8 is to be multiplied by 3, or repeated three times, and is equal to $8+8+8=24$.

The number to be repeated is called the *multiplicand;* the number of times it is to be repeated is called the *multiplier;* and the result is called the *product.* The multiplier and multiplicand are also called *factors* of the product.

Any two numbers which multiplied together produce a third number, are called factors of that number. One number may therefore have many different factors; thus the number 72 may have for factors, 6 and 12, 8 and 9, 3 and 24, since 6×12, 8×9, and 3×24, are each equal to 72.

(10.) Before giving the rule, we will prove 1st, that the product of two numbers is the same in whatever order they are multiplied; and 2d, that where the multiplier can be divided into factors, the product can be obtained by multiplying by these factors successively.

Let the numbers to be multiplied be 5 and 6. It is to be proved that 6 multiplied by 5 is the same as 5 multiplied by 6. Let the figure 1 be written five times in a horizontal line, and let six such lines be written: then considering 5 the multiplicand, we have it repeated six times, and the product equal to 30; but the number of units is the same, whether we consider the five units in the horizontal line repeated six times, or the six units in the vertical line repeated five times; but in this case we should have 6 multiplied by 5.

$$\left.\begin{array}{ccccc}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{array}\right\}\begin{array}{l}3\\ \\2(6\\ \\3\end{array}$$

We may make use of the same diagram to prove the second principle laid down. Let 5 be the multiplicand and 6

the multiplier, and suppose 6 divided into its factors 2 and 3. First we can represent the product of 5 by 3, by writing three lines with five units in each, and then repeat these three lines to represent the multiplication by 2, which gives 5 repeated six times. These same results would be obtained whatever numbers are used, and hence we are authorized to apply the principles laid down to all numbers.

RULE I.

When the multiplier is a single figure.

(11.) *Set the multiplier, under the right hand figure of the multiplicand, and, beginning at the right, multiply each figure of the multiplicand by the multiplier. Set down, in the product, the excess above the tens, and carry the tens as in addition, adding them to the product of the multiplier by the next figure of the multiplicand.*

EXAMPLE.

$$\begin{array}{r} 7835924678 \\ 8 \\ \hline 62687397424 \end{array}$$

Demonstration.—To repeat a number a given number of times is, to repeat the units, tens, hundreds, &c. which it contains; but if we repeat eight units eight times, we shall have sixty four units or six tens and four units. The tens belong to a higher order of units, and therefore must be carried one place to the left, as in addition. The same reasoning will apply to any of the higher orders of units.

RULE II.

When the multiplier contains more than one figure.

(12.) *Multiply the multiplicand by each figure of the multiplier, placing the right hand figure of each product directly under that figure of the multiplier by which it is produced, and take the sum of all the products.*

NOTE.—When there are 0's in either of the factors, since the product of 0 by any number whatever, must be 0, it follows, that there can be no partial product to correspond to the place of 0 in the product, unless some number is carried to this place from the preceding product.

When there are 0's at the right of the multiplier, the multiplication commences at the first significant figure, and the 0's must be written at the right of the product.

Demonstration.—The demonstration of the preceding rule will best be understood by reference to an example. Suppose it requiréd to multiply 4322 by 654. When we multiply the units of the multiplicand by 4, the units of the multiplier, the product will be units, and therefore stands in the unit's place: but when we multiply 2 by 5 tens, or, which is the same thing, (10) multiply 5 tens by 2, the product will be tens, and must therefore stand in the ten's place. For a like reason, the product of hundreds by the units, must be hundreds, and must be put in the place of hundreds; and in general, the product of the units by any figure of the multiplier will be of the same order as that figure, and is therefore to be placed directly beneath it in the product.

$$\begin{array}{r}
4322\\
654\\
\hline
17288\\
21610\\
25932\\
\hline
2826588
\end{array}$$

Rule III.

When the multiplier can be divided into factors.

(13.) *Multiply first by one of the factors, and the resulting product by the other, the last will be the product required.*

The reason for this rule will be understood by a reference to Art. (10).

EXAMPLES.

Multiply 8321 by 56.		5723 × 108	
56 = 7 × 8 therefore	8321	5723	
	7	9	108 = 9 × 12
	58247	51507	
	8	12	
Prod.	465976	618084	

Proof of Multiplication.—Multiply the multiplier by the multiplicand, and if the result is the same, it may be inferred that the work is right.

EXAMPLES FOR EXERCISE.

1. Multiply 123456789 by 7. Ans. 864197523.
2. Multiply 273580961 by 23. Ans. 6292362103.
3. Multiply 82164973 by 3027. Ans. 248713373271.
4. Multiply 8496427 by 874359. Ans. 7428927415293.
5. Multiply 390720400 by 406000. Ans. 158632482400000.
6. Multiply 8057069 by 70050. Ans. 564397683450.
7. Multiply 3161471 by 10000. Ans. 31614710000.
8. Multiply 29753804 by 72. Ans. 2142273888.
9. Multiply 31704592 by 36. Ans. 1141365312.
10. Multiply 160430800 by 108. Ans. 17326526400.

DIVISION.

(14.) Division consists in finding how many times one number is contained in another; or, how many times one number may be subtracted from another. Hence it bears the same relation to subtraction, that multiplication does to addition.

Were it required to find how many times 6 is contained in 24, we might proceed by subtracting 6 from 24 successively, till nothing remained, thus:

24
6 1st subtraction.

18
6 2d subtraction.

12
6 3d subtraction.

6
6 4th subtraction.

0

We see, therefore, that 6 is contained four times in twenty-four, since it can be subtracted from it four times. Division may also be considered the reverse of multiplication, for, as 4 × 6 is equal to 24, it must follow, that 6 is contained four times in twenty four.

The number to be divided is called the *dividend*, the number to divide by is called the *divisor*, and the number of times it is contained in the dividend, is called the *quotient*. If the dividend does not contain the divisor an exact number of times, what remains after the division is completed, is called the *remainder*. The sign of division is ÷, or a horizontal line with the dividend written above and the divisor below; $8 \div 4$ or $\frac{8}{4}$ signifies that 8 is to be divided by 4.

RULE.

(15.) *Write the divisor on the left of the dividend, and draw a curved line between them; draw also a curved line on the right of the dividend to separate it from the quotient.*

Then beginning at the left, find how many times the divisor is contained in as many of the left hand figures of the dividend as are necessary to contain it once, and place the result in the quotient; multiply the divisor by this quotient, and write the product under the figures of the dividend before mentioned.

Subtract this product from that part of the dividend under which it stands, and bring down the next figure of the dividend to the right of the remainder. Divide this last number by the divisor, in the same manner as before, *and continue the operation till all the figures of the dividend are brought down.*

NOTE.—If the product of the divisor by the quotient is greater than the number from which it is to be subtracted, the quotient is too large, and must be diminished; and if the remainder, after subtraction, is greater than the divisor, the quotient is too small, and must be increased.

EXAMPLE.

Divide 937843 by 81.

```
            Dividend.
Divisor 81)937843(11578 quotient.
           81
           ---
           127
            81
           ---
            468
            405
            ---
             634
             567
             ---
              673
              648
              ---
               25 remainder.
```

Demonstration of the Rule.—The number of times one number is contained in another, is evidently equal to the number of times it is contained in the highest order of units in that number, plus the number of times it is contained in all the lower orders of units contained in that number; for example, the number 4 is contained in 8484, 2121 times, that is, 2000 times in 8000, 100 times in 400, 20 times in 80, and once in 4. This may be shown by dividing the parts of 8484 separately, thus:

$$\begin{array}{r} 8000 \div 4 = 2000 \\ 400 \div 4 = 100 \\ 80 \div 4 = 20 \\ 4 \div 4 = 1 \\ \hline 8484 \div 4 = 2121 \end{array}$$

Adding together the results of these separate divisions, we have $8484 \div 4 = 2121$

In this example, where the divisor is contained without remainder in each of the orders of units which compose the dividend, the division might commence either at the right or left of the dividend; but when this is not the case, the division must of necessity commence at the highest order of units. To illustrate this, take 7852 for a dividend, and 4 for a divisor. The highest order of units in the dividend is thousands, and the number might be written, 7000+800+50+2. Now, 4 is contained 1000 times in 7000, and there is a remainder of 3000, in which it is contained only some hundred times. It is also contained some hundred times in 800, the hundreds of the dividend; then in order to find the hundreds of the quotient, we must add the 3000 remainder and the hundreds of the dividend together; their sum is 3800, in which 4 is contained 900 times, with a remainder of 200, which must be added to the tens in order to get the tens of the quotient; and, in like manner, the remainder, arising from the division of the tens must be added to the units, to find the units of the quotient.

The division, as described, may be thus represented:

$$
\begin{array}{rrl}
\text{Divisor.} & \text{Dividend.} & \qquad\qquad\qquad \text{Quotient.} \\
4) & 7000 & +800+50+2(1000+900+60+3=1963 \\
 & 4000 & \\ \cline{2-2}
 & 3000 & \text{1st remainder.} \\
 & 800 & \text{the hundreds.} \\ \cline{2-2}
 & 3800 & \\
 & 3600 & \\ \cline{2-2}
 & 200 & \text{2d remainder.} \\
 & 50 & \text{the tens.} \\ \cline{2-2}
 & 250 & \\
 & 240 & \\ \cline{2-2}
 & 10 & \text{3rd remainder.} \\
 & 2 & \text{the units.} \\ \cline{2-2}
 & 12 & \\
 & 12 & \\ \cline{2-2}
\end{array}
$$

Or thus by the rule:

$$
\begin{array}{l}
4)7852(1963 \\
\;\;4 \\
\hline
\;\;38 \\
\;\;36 \\
\hline
\;\;\;\;25 \\
\;\;\;\;24 \\
\hline
\;\;\;\;\;\;12 \\
\;\;\;\;\;\;12
\end{array}
$$

The only difference in these methods is this:—In the first, the different orders of units are written separately, and in the second, they are written according to the principles of the decimal notation, each figure taking its value from the place which it occupies; and in adding the next lower order of units to the remainder, resulting from the division of a higher order, it is written at the right of that remainder, which is its proper place in the decimal notation. For example; after the first division I find a remainder of 3, which stands in the place of thousands; and in order to add 8 hundreds to this, I have only to write 8 in the next place to the right, which gives 3 thousand 8 hundred, or 38 hundreds. The same principle will apply in passing from any order of units to the next lower; and therefore we may conclude that the rule can be applied to all numbers.

Proof of Division.—Multiply the quotient by the divisor; and if there be a remainder, add it to the product; the result will equal the dividend.

EXAMPLES.

1. Divide 570196382 by 12 Ans. $47516365\frac{2}{12}$.
2. Divide 137896254 by 97 Ans. $1421610\frac{84}{97}$.
3. Divide 35821649 by 764 Ans. $46886\frac{745}{764}$.
4. Divide 4637054283 by 57606 Ans. $80496\frac{1707}{57606}$.

NOTE.—When there is a remainder, it should be written at the right of the quotient with the divisor beneath it.

Contractions in Division.

RULE I.

When the divisor consists of a single figure.

(16.) *Set down the number as before directed, and draw a line beneath the dividend. See how often the divisor is contained in as many of the left hand figures of the dividend as will contain it once, and place the quotient below the dividend; if there be a remainder, consider it as written before the next figure of the dividend, and divide as before; and continue the operation in the same manner through the rest of the dividend.*

This rule is but an abbreviated method of dividing by the preceding rule; the multiplication of the divisor by the quotient, and the subtraction of this product from the dividend, being performed mentally, instead of being written out at length.

EXAMPLES.

Divide 8834679 by 3,

3)8834679

2944893 the quotient.

Divide 973624793 by 7. Ans. 139089256$\frac{1}{7}$.
Divide 17346235782 by 8. Ans. 2168279472$\frac{6}{8}$.
Divide 936237021 by 9. Ans. 104026335$\frac{6}{9}$.
Divide 1333445563 by 4. Ans. 333361390$\frac{3}{4}$.

RULE II.

When there are 0's at the right of the divisor.

(17.) *Cut off the 0's from the divisor, and an equal number of figures from the right of the dividend, and divide with the remaining figures according to the preceding rules; if there be a remainder, write the figures cut off from the dividend on the right of this remainder; this will give the true remainder.*

EXAMPLE.

Divide 3976478 by 63000.

63)000)3976(478(63$\frac{7478}{63000}$
378

196
189

7478

The reasons for this rule will best be understood by a reference to the proof of division by multiplication. For, since the dividend, when it can be exactly divided, is equal to the product of the quotient by the divisor, it will follow that there can be no figures of a lower order in such a dividend than there is in the divisor; for if the last significant figure in the divisor be tens, the product of the divisor by the quotient can contain no significant figure less than tens (12). If the divisor have none less than hundreds or thousands, then the product can have none less than hundreds or thousands. And therefore, if there be significant figures in the dividend of a less order of units than in the divisor, they must belong to the remainder; inasmuch as no quotient whatever, multiplied into the divisor, could produce them.

In the preceding example, the product of 63,000 by any quotient whatever, can contain no figure of a lower order than thousands; and hence the number 478 cut off from the right hand of the divisor, will remain after the division is completed. Moreover, in dividing the remaining figures, we are dividing thousands by thousands, and the remainder 7 is, therefore, 7 thousands, and to this 478 is to be added to make up the whole remainder, and this is done by writing 478 at the right of the remainder.

NOTE.—To divide by 10, 100, 1000, 10,000, or any number consisting of unity and a number of 0's, it is only necessary to cut off from the dividend a number of figures equal to the number of 0's in the divisor.

EXAMPLES.

Divide 137272 by 7800.	Ans. 17$\frac{4672}{7800}$.
Divide 236721 by 20000.	Ans. 11$\frac{16721}{20000}$.
Divide 1783426 by 10000.	Ans. 178$\frac{3426}{10000}$.
Divide 1932147 by 2200.	Ans. 878$\frac{547}{2200}$.

CHAPTER II.

FRACTIONS.

(18.) A *fraction* is an expression to denote a part, or parts of an unit.

It is expressed by two numbers, placed one below the other, with a horizontal line between them. The upper number is called the *numerator*, and the lower the *denominator*. They are also called the *terms* of the fraction.

$\frac{2}{3}$, $\frac{5}{8}$, $\frac{1}{5}$, read, two-thirds, five-eighths, one-fifth, are fractions.

The *denominator* of the fraction denotes the number of *parts* into which the unit is divided, and the *numerator* the number of *parts* expressed by the fraction.

Thus, if we consider one foot the unit of measure, the expression $\frac{4}{5}$ of a foot signifies, that one foot is conceived to be divided into five equal parts, and that the line expressed by the fraction, contains four of these parts.

Fractions may also be considered *unexecuted divisions*, or as resulting from the remainders after division. Suppose it were required to divide a line nineteen feet in length into five equal parts. By dividing 19 by 5, we should have the value of each of these parts. But this division cannot be executed in whole numbers; for, since 5 is contained more than three times, and less than four times, in 19, the length of each part is more than three feet, and less than four feet.

The excess of each of these parts above three feet, must, therefore, be expressed by a fraction. Having divided 19 by 5, we have 3 for a quotient, and a remainder of 4. We may now conceive each foot in this remainder to be divided into five equal parts; and since there are four feet in the remainder, we should have four of these parts, or $\frac{4}{5}$ of a foot for the excess of the fifth part of nineteen feet above three feet. The expression for this part would, therefore, be $3\frac{4}{5}$ feet. The fifth part of nineteen feet might also be written $\frac{19}{5}$, which would signify, that each foot is divided into five parts, and nineteen of these parts are expressed by the fraction. This expression is equivalent to the former, for nineteen fifth parts is the same as 3 and $\frac{4}{5}$.

(19.) A *proper fraction*, is one which has its numerator less than its denominator, and is, consequently, less than unity.

An *improper fraction*, has its numerator greater than its denominator, and is therefore greater than unity.

If the numerator is equal to the denominator, the fraction is equal to unity; for, in this case, the fraction expresses all the parts into which unity is divided.

$\frac{2}{3}$, $\frac{3}{4}$, $\frac{11}{12}$, and $\frac{113}{324}$, are proper fractions.

$\frac{3}{2}$, $\frac{8}{3}$, $\frac{17}{9}$ and $\frac{224}{151}$, are improper fractions.

A *mixed number*, is composed of a whole number and a fraction together, as, $3\frac{4}{5}$, $2\frac{1}{2}$, $7\frac{5}{9}$, &c.

A *compound fraction*, is the fraction of a fraction, $\frac{3}{4}$ of $\frac{5}{8}$, $\frac{2}{3}$ of $\frac{3}{5}$, and $\frac{1}{2}$ of $\frac{7}{8}$, are compound fractions.

A *complex fraction*, is one which has a fraction for its numerator, or denominator, or both; $\frac{2}{\frac{5}{9}}$, $\frac{\frac{3}{5}}{7}$, $\frac{\frac{2}{3}}{\frac{5}{8}}$, are complex fractions.

An *integer*, is a unit or whole number, and is so called as distinguished from a *fraction*, which is a part, or parts, of an unit.

A *prime number*, is one which is not divisible by any number greater than unity; 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, &c., are *prime numbers*.

Two numbers are said to be *prime to each other*, when they have no common divisor greater than unity: thus, 25 and 32 are prime to each other.

The *prime factors* of a number, are such prime numbers as when multiplied together will produce that number, thus: 3, 5 and 2 are the prime factors of 30; 2, 3 and 2, the prime factors of 12; and 2, 2, 2, and 2, the prime factors of 16.

MULTIPLICATION AND DIVISION OF FRACTIONS.

To multiply a fraction by a whole number.

RULE.

(20.) *Multiply the numerator of the fraction, or divide the denominator, by the given number.*

Demonstration of the Rule.—Since the numerator denotes the number of parts expressed by the fraction; as many times as the numerator is repeated, so many times the value of the fraction is increased; and therefore, to multiply the numerator of a fraction by any number, multiplies the fraction by that number; thus, three times $\frac{3}{11}$ is evidently $\frac{9}{11}$, for the value of the parts remaining the same, three times as many parts are expressed by the latter fraction.

Dividing the denominator of a fraction by a whole number also multiplies the fraction by that number: for since the denominator of a fraction expresses the number of parts into which the unit is divided, if the denominator be divided by any number, it will be as many times less, as that number contains units; and hence the value of the parts expressed by the denominator, just so many times greater; and consequently the value of the fraction will be repeated as many times as there are units in the given number. Thus, $\frac{3}{8} \times 2 = \frac{3}{4}$, for $\frac{3}{4}$ is double $\frac{3}{8}$, inasmuch as the number of parts expressed is the same, and the value of each part in the first, is double that of each part in the second.

EXAMPLES.

1. Multiply $\frac{5}{7}$ by 8. Ans. $\frac{40}{7}$.
2. Multiply $\frac{1}{12}$ by 6. Ans. $\frac{6}{12}$ or $\frac{1}{2}$.
3. Multiply $\frac{17}{9}$ by 3. Ans. $\frac{17}{3}$.
4. Multiply $\frac{18}{162}$ by 9. Ans. $\frac{18}{18}$ or 1.
5. Multiply $\frac{316}{716}$ by 4. Ans. $\frac{316}{179}$.
6. Multiply $\frac{51}{32}$ by 9. Ans. $\frac{459}{32}$.

NOTE.—When a fraction is multiplied by a number equal to its denominator, the numerator becomes a whole number, for in that case the denominator becomes unity:

$$\text{Thus, } \frac{3}{5} \times 5 = \frac{15}{5} = 3,\ \frac{17}{12} \times 12 = \frac{17}{1} = 17.$$

To divide a fraction by a whole number.

RULE.

(21.) *Divide the numerator, or multiply the denominator of the fraction by the given number.*

Demonstration of the Rule.—Since the numerator expresses the value of the fraction, to divide the numerator, while the denominator remains the same, is to divide the fraction; and since the denominator shows the number of parts into which the unit is divided, it is evident, that as many times as the denominator is repeated, just so many times less each part becomes; and hence, the numerator remaining the same, the fraction is divided by multiplying the denominator.

Thus $\frac{6}{8} \div 2 = \frac{3}{8}$, or $\frac{6}{16}$, and $\frac{3}{4} \div 2 = \frac{3}{8}$.

EXAMPLES.

1. Divide $\frac{4}{11}$ by 2. Ans. $\frac{2}{11}$ or $\frac{4}{22}$.
2. Divide $\frac{3}{4}$ by 3. Ans. $\frac{1}{4}$ or $\frac{3}{12}$.
3. Divide $\frac{13}{5}$ by 8. Ans. $\frac{13}{40}$.
4. Divide $\frac{371}{411}$ by 6. Ans. $\frac{371}{2466}$.
5. Divide $\frac{33}{52}$ by 11. Ans. $\frac{3}{52}$ or $\frac{33}{572}$.
6. Divide $\frac{144}{233}$ by 12. Ans. $\frac{12}{233}$.

To multiply a fraction by a fraction.

RULE.

(22.) *Multiply the numerator of the multiplicand by the numerator of the multiplier, and the denominator of the multiplicand by the denominator of the multiplier; the resulting fraction will be the product.*

Demonstration.—In multiplying the numerator of the multiplicand by the numerator of the multiplier, the fraction is multiplied (20) by a number as many times too large, as the denominator of the multiplier contains units; and hence it is necessary to divide the product by this denominator; and this division is performed (21) by multiplying the denominator of the multiplicand.

Thus, in multiplying $\frac{3}{7}$ by $\frac{4}{5}$, the product of $\frac{3}{7}$ by 4 is $\frac{12}{7}$, which is five times too large; for each of the units in 4 is but the fifth part of the unit of which $\frac{4}{5}$ is a fraction; the product $\frac{12}{7}$ must, therefore, be divided by 5, which is done by multiplying the denominator, and the result is $\frac{12}{35}$.

EXAMPLES.

1. Multiply $\frac{3}{4}$ by $\frac{9}{11}$. Ans. $\frac{27}{44}$.
2. Multiply $\frac{5}{8}$ by $\frac{2}{3}$. Ans. $\frac{10}{24}$.
3. Multiply $\frac{17}{63}$ by $\frac{11}{12}$. Ans. $\frac{187}{756}$.
4. Multiply $\frac{14}{15}$ by $\frac{7}{8}$. Ans. $\frac{98}{120}$.
5. Multiply $\frac{172}{379}$ by $\frac{63}{45}$. Ans. $\frac{10836}{17055}$.

To divide a fraction by a fraction.

RULE.

(23.) *Multiply the denominator of the dividend by the numerator of the divisor, and the numerator of the dividend by the denominator of the divisor; or, which is the same thing, invert the divisor, and proceed as in multiplication.*

Demonstration.—In multiplying the denominator of the dividend by the numerator of the divisor (21), the dividend is divided by a number as many times too large as the denominator of the divisor contains units; and it is necessary to multiply by this denominator to arrive at the true quotient; this is done by multiplying the numerator of the dividend (20).

Thus, in dividing $\frac{7}{8}$ by $\frac{3}{4}$, I begin by dividing $\frac{7}{8}$ by 3, which gives $\frac{7}{24}$ for a quotient; but this quotient is too small, for in dividing by 3, I divided by a number four times too large, each of the units in 3 being only $\frac{1}{4}$ of the unit of which $\frac{3}{4}$ is a fraction. I must, therefore, multiply this quotient by 4 to arrive at the true quotient; and there results the fraction $\frac{28}{24}$.

EXAMPLES.

1. Divide $\frac{25}{9}$ by $\frac{3}{5}$.	Ans. $\frac{125}{27}$.
2. Divide $\frac{6}{7}$ by $\frac{1}{4}$.	Ans. $\frac{24}{7}$.
3. Divide $\frac{1}{8}$ by $\frac{31}{26}$.	Ans. $\frac{20}{248}$.
4. Divide $\frac{17}{31}$ by $\frac{11}{16}$.	Ans. $\frac{272}{341}$.
5. Divide $\frac{5}{6}$ by $\frac{15}{7}$.	Ans. $\frac{35}{90}$.
6. Divide $\frac{2}{7}$ by $\frac{3}{5}$.	Ans. $\frac{10}{21}$.
7. Divide $\frac{1}{8}$ by $\frac{1}{16}$.	Ans. $\frac{16}{8}$, or 2.
8. Divide $\frac{31}{22}$ by $\frac{11}{12}$.	Ans. $\frac{372}{242}$.

REDUCTION OF FRACTIONS.

(24.) Two fractions may be equal to each other when the terms of those fractions are unequal; for example, $\frac{6}{8}$ is equal to $\frac{3}{4}$; for the denominator of the first fraction being double that of the second, the parts of unity expressed by the first have but half the value of those expressed by the second; and hence, six of the former are equal to three of the latter. A fraction may be expressed in a variety of forms, thus: $\frac{1}{2}$, $\frac{2}{4}$, $\frac{4}{8}$, $\frac{8}{16}$, are all equal; for, as the number of parts expressed by the fraction increases, the value of the parts decreases.

A fraction is said to be reduced to its lowest terms when, the value remaining the same, its numerator and denominator are the least numbers that will express that value. Thus, the fraction $\frac{6}{8}$, reduced to its lowest terms, is $\frac{3}{4}$, for its value cannot be expressed by less numbers.

To reduce a fraction to its lowest terms.

RULE.

(25.) ***Divide the numerator and denominator by any number that will divide both without remainder, and divide these quotients in the same manner, till it is found that there is no number greater than unity that will divide them. The fraction will then be in its lowest terms.***

Demonstration of the Rule.—By dividing the numerator and denominator of a fraction by the same number, the value of the fraction will not be altered, since the fraction is divided by dividing its numerator (21), and multiplied by dividing its denominator (20). The fraction, therefore, having been first divided, and afterwards multiplied by the same number, its value is not altered.

EXAMPLES.

1. Reduce $\frac{216}{288}$ to its lowest terms.*

$$\begin{array}{ccccccc} \text{1st Div.} & & \text{2d Div.} & & \text{3d Div.} & & \\ (4.) & & (3\,) & & (6.) & & \\ \frac{216}{288} & = & \frac{54}{72} & = & \frac{18}{24} & = & \frac{3}{4} \text{ the answer.} \end{array}$$

---

* The following rules will be useful in reducing fractions by this rule.

1st. Every number ending in an even number or 0 is divisible by 2.

2d. Every number ending in 5 or 0, is divisible by 5.

3d. Every number ending in 0, is divisible by 10; in 00, by 100; in 000, by 1000

4th. If the sum of the figures, expressing any number, be divisible by 3 or 9, the number itself is divisible by 3 or 9.

5th. If the right hand figure be even, and the sum of the digits be divisible by 6, then the number itself is divisible by 6.

6th. When numbers connected by the sign of addition or subtraction are to be divided by any number, each of those numbers must be divided by it.

$$\text{Thus: } \frac{10+8-4}{2} = 5+4-2.$$

7th. But if the numbers are connected by the sign of multiplication, only one is to be divided, thus:

$$10 \times 8 \times 3 \div 2 = 5 \times 8 \times 3, \text{ and } 5 \times 8 \times 3 \div 4 = 5 \times 2 \times 3.$$

2. Reduce $\frac{195}{780}$ to its lowest terms.

(5.) (3.) (13.)

$\frac{195}{780} = \frac{39}{156} = \frac{13}{52} = \frac{1}{4}$ the answer.

3. Reduce $\frac{136}{204}$ to its lowest terms. Ans. $\frac{2}{3}$.

4. Reduce $\frac{525}{630}$ to its lowest terms. Ans. $\frac{5}{6}$.

5. Reduce $\frac{231}{252}$ to its lowest terms. Ans. $\frac{11}{12}$.

Rule II.

Divide the terms of the fraction by the greatest number that will divide both without remainder, and the fraction will be reduced to its lowest terms by a single division.

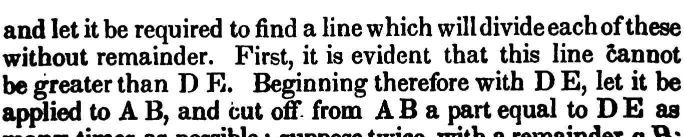
The greatest number that will divide two numbers without remainder, is called the *greatest common divisor* of those numbers; and the application of the preceding rule involves the following problem :—

To find the greatest common divisor of two numbers.

Rule.

(27.) *Divide the greater by the less, then divide the divisor by the remainder, and continue the operation in the same manner, always dividing the last divisor by the last remainder, till the division can be executed without remainder. The last divisor will be the greatest common divisor of the two numbers.*

Demonstration.—To make the demonstration of this rule general, let any two numbers be represented by the lines A B and D E;



and let it be required to find a line which will divide each of these without remainder. First, it is evident that this line cannot be greater than D E. Beginning therefore with D E, let it be applied to A B, and cut off from A B a part equal to D E as many times as possible; suppose twice, with a remainder a B;

then cut off from DE a part equal to this remainder aB, as many times as possible; suppose three times, with a remainder cE; from the first remainder aB, cut off a part equal to cE, as many times as possible; suppose twice with a remainder dB; from the second remainder cE, cut off a part equal to dB, as many times as possible, and continue this process till a remainder is found, that will exactly divide the preceding one; this last remainder will be the greatest common divisor of the two lines.

For, let us suppose that the third remainder, dB, will divide the second, cE, without remainder; it will also divide aB, for aB is equal to twice $cE + dB$; and since it will divide both aB and cE, it will also divide DE, which is equal to three times $aB + cE$; and it being proved that dB divides both aB and DE without remainder, it will also divide AB, which is equal to twice $DE + aB$; and hence dB is the *common divisor* of the lines AB and DE; and as we have taken, in each division, the greatest line which would divide both lines, it is the *greatest* common divisor. This demonstration is independent of the length of the given lines, and consequently will apply to all numbers whatever.

To apply this demonstration to a particular example, let it be required to find the greatest common divisor of 204 and 468.

$$\begin{array}{l}
204)468(2 \quad \text{1st division.} \\
\quad\;\, 408 \\
\hline
\qquad 60)204(3 \quad \text{2d division.} \\
\qquad\quad\;\, 180 \\
\hline
\qquad\qquad 24)60(2 \quad \text{3d division.} \\
\qquad\qquad\quad\; 48 \\
\hline
\qquad\qquad\quad 12)24(2 \quad \text{4th division.} \\
\qquad\qquad\qquad\; 24 \\
\hline
\qquad\qquad\qquad\; 00
\end{array}$$

It is evident, in the first place, that there can be no greater common divisor of the two numbers than 204, the less number. I therefore divide by 204, which is contained twice in 468, with a remainder of 60; and the problem is now reduced to finding the greatest common divisor of 204 and 60; for the number that will divide both 204 and 60, will also divide 468, which is equal to $2 \times 204 + 60$.

By the next division, the problem is reduced to finding the greatest common divisor of 24 and 60; and by the last division to finding the greatest common divisor of 12 and 24; and since 12 will divide both itself and 24 without remainder, it is the greatest common divisor of these two numbers, and also of the given numbers, 204 and 468. For, since 12 divides 24, it will also divide $2 \times 24 + 12$, or $48 + 12 = 60$; and as it divides without remainder both 24 and 60, it will also divide $3 \times 60 + 24$, or $180 + 24 = 204$; and dividing both 60 and 204, it will also divide, without remainder, $2 \times 204 + 60$, or $408 + 60 = 468$; and therefore, 12 is the greatest common divisor of 204 and 468.

EXAMPLES.

1. Find the greatest common divisor of 1908 and 936. Ans. 36.
2. Find the greatest common divisor of 246 and 372. Ans. 6.
3. Find the greatest common divisor of 324 and 612. Ans. 36.
4. Find the greatest common divisor of 327 and 932.
5. Find the greatest common divisor of 462 and 743.

EXAMPLES IN REDUCTION OF FRACTIONS.

1. Reduce $\frac{299}{391}$ to its lowest terms. Ans. $\frac{13}{17}$.
2. Reduce $\frac{453}{1057}$ to its lowest terms. Ans. $\frac{3}{7}$.
3. Reduce $\frac{6291}{6757}$ to its lowest terms. Ans. $\frac{27}{29}$.
4. Reduce $\frac{5389}{8364}$ to its lowest terms. Ans. $\frac{317}{492}$.

To reduce a whole number to a fraction having a given denominator.

RULE.

(28.) *Multiply the whole number by the given denominator, and write the product with the given denominator beneath it.*

Demonstration.—A whole number may be considered a fraction having unity for its denominator; as the quotient of any number divided by unity is equal to the number itself. Hence, if we would reduce a whole number to a fraction having a given denominator, we have only to write it in a fractional form with unity for its denominator, and multiply both terms by the given denominator. For example, let 4 be reduced to a fraction whose denominator shall be 3 : 4 is the same as $\frac{4}{1}$, and multiplying both terms by 3, we have $\frac{12}{3} = \frac{4}{1}$ or 4. The fraction $\frac{4}{1}$ having been both multiplied and divided by 3, the value remains the same. This method is evidently the same as multiplying 4 by 3, and writing 3 beneath the product.

Examples.

1. Reduce 7 to a fraction whose denominator shall be 9.

$7 \times 9 = 63$. $\frac{63}{9}$ the required fraction.

2. Reduce 12 to a fraction whose denominator shall be 7.

$$\frac{12 \times 7}{1 \times 7} = \frac{84}{7} \text{ or } 7 \times 12 = 84. \quad \frac{84}{7} \text{ the answer.}$$

To reduce a mixed number to an improper fraction.

Rule.

(29.) *Multiply the whole number by the denominator of the fractional part, and to the product add the numerator, and write the sum over the denominator.*

This rule depends upon the same principle as the preceding ; the whole number being multiplied and divided by the denominator of the fractional part, its value will not be altered.

Examples.

1. Reduce $23\frac{2}{5}$ to an improper fraction.

$$23 \times 5 = \frac{115+2}{5} = \frac{117}{5}, \text{ the answer.}$$

2. Reduce $12\frac{7}{9}$ to an improper fraction.

Ans. $\frac{115}{9}$.

3. Reduce $131\frac{2}{3}$ to an improper fraction.

Ans. $\frac{395}{3}$.

4. Reduce $67\frac{4}{11}$ to an improper fraction.

Ans. $\frac{741}{11}$.

To reduce an improper fraction to a whole or mixed number.

RULE.

(30.) *Divide the numerator by the denominator, and the quotient will be the whole or mixed number sought.*

EXAMPLES.

1. Reduce $\frac{27}{4}$ to a whole or mixed number.
Ans. $27 \div 4 = 6\frac{3}{4}$.

2. Reduce $\frac{178}{18}$ to a whole or mixed number.
Ans. $9\frac{16}{18}$.

3. Reduce $\frac{738}{18}$ to a whole or mixed number.
Ans. 41.

4. Reduce $\frac{7321}{17}$ to a whole or mixed number.
Ans. $430\frac{11}{17}$.

5. Reduce $\frac{131}{9}$ to a whole or mixed number.
Ans. $14\frac{5}{9}$.

To reduce a compound fraction to a simple one.

RULE.

(31.) *Multiply all the numerators together for a numerator, and all the denominators for a denominator; the result will be the simple fraction required.*

Demonstration.—To show the correctness of this rule, let it be required to reduce the compound fraction, $\frac{2}{3}$ of $\frac{5}{8}$, to a simple one; $\frac{2}{3}$ of $\frac{5}{8}$ is the same as twice $\frac{1}{3}$ of $\frac{5}{8}$; to take $\frac{1}{3}$ of $\frac{5}{8}$, we should divide $\frac{5}{8}$ by 3; $\frac{5}{8} \div 3 = \frac{5}{24}$, and multiplying this result by 2, we have $\frac{10}{24}$, or $\frac{5}{8} \times \frac{2}{3} = \frac{10}{24}$. It is evident that the same principle would apply to any compound fraction whatever. When the compound fraction consists of more than two single ones, we may at first reduce two of them, and afterwards reduce the resulting fraction with the next, and so on. This would be the same as multiplying all the numerators and all the denominators together.

EXAMPLES.

1. Reduce $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{7}{8}$ to a simple fraction.

$$\frac{1\times3\times7}{2\times4\times8}=\frac{21}{64},\text{ the simple fraction required.}$$

2. Reduce $\frac{7}{8}$ of $\frac{9}{2}$ of $\frac{1}{6}$ to a simple fraction.

Ans. $\frac{63}{96}=\frac{21}{32}$.

3. Reduce $\frac{1}{3}$ of $\frac{3}{4}$ of 1 to a simple fraction.

Ans. $\frac{1}{4}$.

4. Reduce $\frac{5}{6}$ of $13\frac{1}{3}$ of 8 to a simple fraction.

Ans. $\frac{5}{6}\times\frac{40}{3}\times\frac{8}{1}=\frac{1600}{18}=\frac{800}{9}$.

To reduce a complex fraction to a simple one.

RULE.

(32.) *If the denominator be a fraction, multiply the numerator of the given fraction by the denominator of the denominator; if the numerator be a fraction, multiply its denominator by the denominator of the given fraction; if both numerator and denominator are fractions, multiply the numerator of the numerator by the denominator of the denominator, and the denominator of the numerator by the numerator of the denominator.*

For an application of the rule, take the complex fractions, $\frac{3}{\frac{3}{5}}$, $\frac{\frac{4}{7}}{9}$ and $\frac{\frac{3}{5}}{\frac{7}{9}}$. The first has a fractional denominator, and reduced, by the rule, gives $\frac{15}{3}$; the second has a fractional numerator, and reduced is $\frac{4}{63}$; and the third, in which both terms are fractional, becomes $\frac{3\times9}{5\times7}=\frac{27}{35}$.

Demonstration of the Rule.—Considering fractions as unexecuted divisions, we shall have in a complex fraction the denominator of which is fractional, the numerator to be divided by a fraction, which is done by inverting the divisor and multiplying (23). This is the same as multiplying the numerator by the denominator of the fractional part. Thus, $\frac{3}{\frac{3}{5}}$ might be written $3\div\frac{3}{5}$, or (23) $3\times\frac{5}{3}=\frac{15}{\text{[illegible]}}$. When the numerator is fractional, it denotes that a fraction is to be divided by a whole number, which is done by multiplying its denominator (21); $\frac{\frac{4}{7}}{9}$ may be written $\frac{4}{7} \div 9$, which becomes, by performing the division, indicated $\frac{4}{63}$. When both terms of the complex fraction are fractional, we have a fraction to be divided by a fraction; $\frac{\frac{3}{5}}{\frac{7}{9}}$ might be written $\frac{3}{5} \div \frac{7}{9}$, or (23) $\frac{3}{5} \times \frac{9}{7} = \frac{27}{35}$. This operation, though written in a different form, is evidently the same as that given in the rule.

EXAMPLES.

1. Reduce $\frac{57}{\frac{41}{3}}$ to a simple fraction.

Ans. $\frac{57 \times 3}{41} = \frac{171}{41}$.

2. Reduce $\frac{6\frac{1}{4}}{3}$ to a simple fraction.

Ans. $\frac{6\frac{1}{4}}{3} = \frac{\frac{25}{4}}{3} = \frac{25}{12}$.

3. Reduce $\frac{5}{1\frac{1}{2}}$ to a simple fraction.

Ans. $\frac{10}{3}$.

4. Reduce $\frac{5\frac{7}{8}}{6\frac{9}{10}}$ to a simple fraction.

Ans. $\frac{5\frac{7}{8}}{6\frac{9}{10}} = \frac{\frac{47}{8}}{\frac{69}{10}} = \frac{470}{552}$.

5. Reduce $\frac{2\frac{1}{2}}{3\frac{1}{4}}$ to a simple fraction.

Ans. $\frac{20}{26}$ or $\frac{10}{13}$.

6. Reduce $\frac{92}{\frac{35}{3}}$ to a simple fraction.

Ans. $\frac{276}{35}$.

To reduce the lower denominations of a compound number, to a fraction of a higher denomination.

RULE.

(33.) *Reduce the given quantity to the lowest denomination mentioned, and reduce an integer of the higher denomination to the same. The latter will be the denominator, and the former the numerator, of the proposed fraction.*

EXAMPLES.

1. Reduce 16 s. . 4 d. . 2 qrs. to the fraction of a pound.

	1
	20
16 . 4 . 2	20
12	12
196	240
4	4
786 numerator.	960 denominator.

$\frac{786}{960} = \frac{131}{160}$, the fraction required.

2. Reduce 4 hours, 16 minutes, and 12 seconds, to the fraction of a day.

Ans. $\frac{15372}{86400} = \frac{427}{2400}$.

3. Reduce 3 furlongs, 36 rods, to the fraction of a mile.

Ans. $\frac{39}{80}$.

4. Reduce 2 yards, 1 foot, 6 inches, to the fraction of a rod.

Ans. $\frac{5}{11}$.

To find the value of a fraction of a higher denomination, in whole numbers of lower denominations.

RULE.

(34.) *Multiply the numerator of the fraction by the parts in the next inferior denomination, and divide the product by the denominator. Then if any thing remains, multiply it by the parts in the next inferior denomination, and divide by the denominator as before; and continue the ope-*

ration in the same manner as far as necessary. The several quotients, placed in order, will be the value of the fraction required.

EXAMPLES.

1. What is the value of $\frac{3}{8}$ of a shilling.

```
   3
  12
 ----
8)36(4 d.  2 qrs. the answer.
  32
 ----
   4
   4
 ----
8)16(2 qrs.
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2. What is the value of $\frac{4}{5}$ of a lb. troy?
Ans. 9 oz. 12 dwts.

3. What is the value of $\frac{7}{8}$ of an acre?
Ans. 3 roods 20 rods.

4. What is the value of $\frac{3}{10}$ of a day?
Ans. 7 hours 12 min.

5. What is the value of $\frac{5}{11}$ of a rod?
Ans. 2 yds. 1 ft. 6 in.

To reduce fractions of different denominators to equivalent fractions, having a common denominator.

RULE.

(35.) *Multiply the denominators together for a common denominator, and each numerator into all the denominators, except its own, for new numerators.*

Demonstration.—The demonstration of the preceding rule depends upon the following principle. *If both terms of a fraction be multiplied by the same number, the value of the fraction is not altered;* for, by multiplying the numerator, the fraction is multiplied (20), and by multiplying the denominator, the fraction is divided (21) by the same number.

Let it now be required to reduce $\frac{3}{4}$, $\frac{5}{7}$ and $\frac{2}{3}$ to a common denominator. We have, for the common denominator,

$4\times7\times3=84$, and for the numerators respectively, $3\times7\times3=63$, $5\times4\times3=60$, and $2\times4\times7=56$. The fractions, when reduced to a common denominator, are, therefore, $\frac{63}{84}$, $\frac{60}{84}$ and $\frac{56}{84}$. By writing these results in a different manner, it will be seen, that each fraction has been multiplied in both its terms, by the product of the denominators of the two others. Thus, $\frac{63}{84}=\frac{3\times(7\times3)}{4\times(7\times3)}$, $\frac{60}{84}=\frac{5\times(4\times3)}{7\times(4\times3)}$ and $\frac{56}{84}=\frac{2\times(4\times7)}{3\times(4\times7)}$. The value of each fraction, therefore, is not altered. Whatever be the number of fractions, the same reasoning will apply, for the operation will be equivalent to multiplying the terms of each fraction, by the product of the denominators of all the others.

EXAMPLES.

1. Reduce $\frac{5}{7}$, $\frac{3}{8}$ and $\frac{9}{10}$ to a common denominator.

Ans. $\frac{400}{560}$, $\frac{210}{560}$ and $\frac{504}{560}$.

2. Reduce $\frac{2}{3}$, $\frac{3}{5}$ and $\frac{3}{4}$ to a common denominator.

Ans. $\frac{40}{60}$, $\frac{36}{60}$ and $\frac{45}{60}$.

3. Reduce $\frac{5}{6}$, $2\frac{3}{5}$ and 4 to a common denominator.

Ans. $\frac{25}{30}$, $\frac{78}{30}$ and $\frac{120}{30}$.

4. Reduce $\frac{5}{7}$, $\frac{1}{8}$ and $2\frac{1}{4}$ to a common denominator.

Ans. $\frac{160}{224}$, $\frac{28}{224}$ and $\frac{504}{224}$.

5. Reduce $\frac{\frac{3}{4}}{6}$, $\frac{5}{\frac{2}{3}}$ and $\frac{4}{7}$ of $\frac{1}{3}$ to a common denominator.

Ans. $\frac{126}{1008}$, $\frac{7560}{1008}$ and $\frac{192}{1008}$.

(36.) When the less denominator of two fractions exactly divides the greater, *multiply both terms of that which has the least denominator by the quotient.*

EXAMPLES.

1. Reduce $\frac{3}{7}$ and $\frac{5}{14}$ to a common denominator.

Dividing 14 by 7, the quotient is 2; and if we multiply both terms of the first fraction by 2, it becomes $\frac{6}{14}$, and the two fractions have the common denominator, 14.

2. Reduce $\frac{11}{12}$ and $\frac{141}{288}$ to a common denominator.

Ans. $\frac{264}{288}$ and $\frac{141}{288}$.

3. Reduce $\frac{1}{6}$, $\frac{3}{8}$, $\frac{1}{4}$ and $\frac{12}{24}$ to a common denominator.

Ans. $\frac{4}{24}$, $\frac{9}{24}$, $\frac{6}{24}$ and $\frac{12}{24}$.

4. Reduce $\frac{1}{12}$, $\frac{3}{16}$, and $\frac{17}{96}$ to a common denominator.

Ans. $\frac{8}{96}$. $\frac{18}{96}$ and $\frac{17}{96}$.

To reduce fractions of different denominators to the least common denominator.

(37.) To reduce fractions to their least common denominator, it is necessary to find the least number which can be divided, without remainder, by each of the given denominators. Such a number being found, we might divide it by each denominator, and multiply both terms of the fraction by the quotient. Take, for example, the fractions $\frac{5}{12}$ and $\frac{7}{16}$; 48 is the least number which can be divided by both denominators, dividing 48 by 12, gives 4, for the multiplier of the terms of the first fraction, which, by performing the multiplication, becomes $\frac{5\times4}{12\times4}=\frac{20}{48}$; in like manner we obtain 3 for the multiplier of the second fraction, which becomes $\frac{7\times3}{16\times3}=\frac{21}{48}$.

The least number which can be divided, by several other numbers, without remainder, is called the *least common multiple* of those numbers.

To find the least common multiple of two or more numbers.

RULE.

(38.) *Write the numbers one after the other, and draw a line beneath them; then take any* prime *number which will divide two or more of them without remainder, and divide all the numbers that it will so divide, writing the quotients beneath, and also all the numbers which are not divisible by it. Find a* prime *number which will divide two or more of the numbers of this second line without remainder, and proceed as before; and continue the operation till there are no two numbers left which have a common divisor. Then multiply together all the numbers remaining, and all the divisors; the product will be the least common multiple sought.*

Let it be required to find the least common multiple of 12, 15, 7, 18, 3, 5 and 35.

7	12,	15,	7,	18,	3,	5,	35,
5	12,	15,	1,	18,	3,	5,	5,
3	12,	3,	1,	18,	3,	1,	1,
3	4,	1,	1,	6,	3,	1,	1,
2	4,	1,	1,	2,	1,	1,	1,
	2,	1,	1,	1,	1,	1,	1,

$$7\times5\times3\times3\times2\times2=1260.$$

First, dividing by the prime number 7, I find that it divides two of the given numbers, 7 and 35. The rest being written in the line below, I take 5, for the next divisor, which divides 15 and 5. Continuing the operation in the same manner, I divide successively, by 3, 3 and 2, when the division can be carried no further; then multiplying all the divisors together, and that product by 2, (the only number greater than unity found in the lower line), the result is 1260.

Now, to show that 1260 is a multiple of each of the given numbers, it is only necessary to observe, that among its prime factors, 7, 5, 3, 3, 2, 2, we can find three of the given numbers, namely, 7, 5 and 3; it must, therefore, be divisible by each of these. Moreover, the prime factors of each of the other numbers, are found among the factors of 1260, and, therefore, it is divisible by each of those numbers; for example, $3\times2\times2=12$. And it is evident, that $3\times2\times2\times5\times7\times3$, or, $12\times5\times7\times3=1260$ is divisible by 12; and that $3\times5\times7\times3\times2\times2$, or, $15\times7\times3\times2\times2=1260$ is divisible by 15, and so of the others.

It is also the *least* common multiple of the given numbers; for it has, among its factors, those only, which are necessary to make up the prime factors of the given numbers. It is also evident that the rule will apply to any numbers whatever, since it is nothing more than finding the prime factors of the given numbers, and taking their product.

EXAMPLES.

1. Find the least common multiple of 9, 16, 35, 20 and 12.

2	9,	16,	35,	20,	12,
3	9,	8,	35,	10,	6,
2	3,	8,	35,	10,	2,
5	3,	4,	35,	5,	1,
	3,	4,	7,	1,	1,

$2 \times 3 \times 2 \times 5 \times 3 \times 4 \times 7 = 5040$, the least common multiple.

2. Find the least common multiple of 3, 4, 9, 10, 21, 35 and 12. Ans. 1260.

3. Find the least common multiple of 2, 3, 4, 5, 6, 7, 8 and 9. Ans. 2520.

4. Find the least common multiple of 16, 11, 33, 27 and 45. Ans. 23,760.

To reduce fractions to the least common denominator.

RULE.

(39.) *Find the least common multiple of all the denominators; this will be the common denominator. For the numerators, divide the common denominator by each particular denominator, and multiply the quotient by the numerator.*

EXAMPLES.

1. Reduce $\frac{1}{2}$, $\frac{4}{7}$, $\frac{3}{16}$ and $\frac{2}{21}$ to their least common denominator.

7	2,	7,	16,	21,
2	2,	1,	16,	3,
	1,	1,	8,	3,

$7 \times 2 \times 8 \times 3 = 336$, least common denominator.

	Denom'rs.		Quot's.		Num'rs.		New num'rs.
336 ÷	2	=	168	×	1	=	168
336 ÷	7	=	48	×	4	=	192
336 ÷	16	=	21	×	3	=	63
336 ÷	21	=	16	×	2	=	32

Ans. $\frac{168}{336}$, $\frac{192}{336}$, $\frac{63}{336}$, $\frac{32}{336}$.

2. Reduce $\frac{9}{11}$, $\frac{3}{8}$, $\frac{4}{7}$ and $\frac{1}{4}$, to their least common denominator. Ans. $\frac{504}{616}$, $\frac{231}{616}$, $\frac{352}{616}$, $\frac{154}{616}$.

3. Reduce $\frac{1}{25}$, $\frac{3}{10}$, $\frac{49}{50}$ and $\frac{71}{75}$, to their least common denominator. Ans. $\frac{6}{150}$, $\frac{45}{150}$, $\frac{147}{150}$, $\frac{142}{150}$.

4. Reduce $\frac{4}{9}$, $\frac{2}{21}$, $\frac{3}{4}$ and 8, to their least common denominator. Ans. $\frac{112}{252}$, $\frac{24}{252}$, $\frac{189}{252}$, $\frac{2016}{252}$.

5. Reduce $5\frac{1}{2}$, $7\frac{1}{4}$ and $6\frac{3}{8}$, to their least common denominator. Ans. $\frac{44}{8}$, $\frac{58}{8}$, $\frac{51}{8}$.

6. Reduce $\frac{3}{4}$ of $\frac{1}{2}$, $\frac{4}{\frac{2}{5}}$ and $\frac{\frac{1}{2}}{7}$, to their least common denominator. Ans. $\frac{21}{56}$, $\frac{560}{56}$, $\frac{4}{56}$.

ADDITION OF FRACTIONS.

(40). Before fractions can be added together, they must be reduced to a common denominator, so that the parts of an unit, expressed by each fraction, may be the same (18). For example, the sum of $\frac{1}{3}$ and $\frac{1}{4}$ can neither be expressed in thirds nor fourths. But, since $\frac{1}{3} = \frac{4}{12}$, and $\frac{1}{4} = \frac{3}{12}$, their sum can be expressed in twelfths, and will be equal to $\frac{7}{12}$. Hence we have, for the addition of fractions, the following

RULE.

Reduce the fractions to a common denominator, and write the sum of the numerators over the common denominator.

EXAMPLES.

1. Find the sum of $\frac{3}{4}$, $\frac{2}{3}$ and $\frac{4}{9}$.

These fractions reduced to a common denominator become, $\frac{27}{36}$, $\frac{24}{36}$ and $\frac{16}{36}$.

Taking the sum of the numerators, we have for the sum of the fractions, $\frac{67}{36}$ or, $1\frac{31}{36}$.

2. Find the sum of $\frac{3}{4}$ and $\frac{5}{9}$. Ans. $1\frac{11}{36}$.

3. Find the sum of $\frac{2}{3}$, $\frac{3}{5}$ and $\frac{5}{7}$. Ans. $1\frac{103}{105}$.

4. Find the sum of $\frac{5}{9}$, $\frac{3}{5}$ and $2\frac{1}{5}$,. Ans. $\frac{151}{45}$ or $3\frac{16}{45}$.

When mixed numbers are to be added, they may be reduced to improper fractions; or, the fractional parts may be added separately, and the whole numbers added to the sum.

5. Find the sum of $5\frac{1}{2}$, $3\frac{5}{7}$, $1\frac{1}{8}$ and $\frac{3}{4}$.

Here the sum of the fractional parts is $2\frac{5}{56}$; the sum of the whole numbers is 9; the sum of the whole is, therefore, $11\frac{5}{56}$.

6. Find the sum $3\frac{1}{4}$, $\frac{4}{21}$ $\frac{13}{7}$, $6\frac{1}{3}$ and 28. Ans. $39\frac{53}{84}$.

SUBTRACTION OF FRACTIONS.

RULE.

(41.) *Prepare the fractions as in addition, and write the difference of the numerator over the common denominator.*

EXAMPLES.

1. Subtract $\frac{1}{3}$ from $\frac{3}{5}$.

$\frac{3}{5} = \frac{9}{15}$ and $\frac{1}{3} = \frac{5}{15}$; $\frac{9}{15} - \frac{5}{15} = \frac{4}{15}$, the answer.

2. Subtract $\frac{5}{12}$ from $\frac{7}{13}$. Ans. $\frac{19}{156}$.

3. Subtract $\frac{6}{11}$ from $\frac{7}{12}$. Ans. $\frac{5}{121}$.

4. Subtract $\frac{2}{7}$ of $4\frac{1}{2}$ from $5\frac{3}{8}$. Ans. $4\frac{5}{56}$.

5. Subtract $3\frac{1}{3}$ from $5\frac{7}{8}$. Ans. $2\frac{13}{24}$.

6. Subtract $\frac{\frac{2}{3}}{7}$ from $\frac{1}{\frac{2}{3}}$. Ans. $\frac{59}{42}$ or $1\frac{17}{42}$.

EXAMPLES IN ADDITION AND SUBTRACTION.

1. Find the sum of $\frac{1}{2} - \frac{1}{4} + \frac{3}{5} - \frac{1}{3}$. Ans. $\frac{31}{60}$.

In examples like the preceding, all the fractions having the sign +, are to be collected into one sum, and all those having the sign —, into another, and the difference between these sums taken.

2. Find the sum of $\frac{6}{7} - \frac{3}{4} - \frac{1}{8} + \frac{19}{20} - \frac{11}{12}$. Ans. $\frac{13}{840}$.
3. Find the sum of $\frac{1}{12} + \frac{7}{8} - \frac{3}{4} + \frac{5}{6} - \frac{1}{24}$. Ans. 1.
4. Find the sum of $4\frac{1}{4} + 6\frac{1}{6} - 5\frac{1}{5} - 3\frac{1}{3}$. Ans. $1\frac{53}{60}$.

CONTINUED FRACTIONS.

(42.) A continued fraction is one which has a whole number, and a fraction for its denominator, the fractional part of which, has also a whole number and fraction for its denominator, and so on.

For example,
$$\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{3}}}}$$
is a continued fraction.

A continued fraction is formed from a simple fraction, the terms of which are prime to each other, by dividing both terms by the numerator. The denominator not being exactly divisible by the numerator, will become a mixed number, the fractional part of which is divided as before.

In the example above, the original fraction was $\frac{11}{37}$. Dividing both terms of this fraction by 11, it becomes $\frac{1}{3\frac{4}{11}}$; again dividing both terms of $\frac{4}{11}$ by 4, we obtain $\frac{1}{2+\frac{3}{4}}$; and substituting this in the place of $\frac{4}{11}$, the fraction becomes
$$\cfrac{1}{3+\cfrac{1}{2+\frac{3}{4}}}$$

Operating in the same manner upon $\frac{3}{4}$, we find $\frac{3}{4} = \frac{1}{1+\frac{1}{3}}$, which, being substituted in the place of $\frac{3}{4}$ in the continued fraction, it becomes

$$\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{3}}}}.$$

Here must the division stop, the numerator of the last fraction being unity.

If it be required to reduce a continued fraction to a simple one, we have only to apply the rule for the reduction of complex fractions (32); for, by beginning at the last, and reducing the mixed denominator to an improper fraction, we shall have a series of complex fractions, the denominators of which are fractional. In the example given above, take the last fraction, $\frac{1}{1\frac{1}{3}}$; by reducing the mixed denominator to an improper fraction, it becomes $\frac{1}{\frac{4}{3}} = \frac{3}{4}$ (32), and the continued fraction is reduced to

$$\cfrac{1}{3+\cfrac{1}{2+\frac{3}{4}}}.$$

Taking now the fraction $\frac{1}{2+\frac{3}{4}}$, we find it equal to $\frac{1}{\frac{11}{4}} = \frac{4}{11}$, the fraction then becomes $\frac{1}{3+\frac{4}{11}} = \frac{1}{\frac{37}{11}} = \frac{11}{37}$, the simple fraction required.

In this way, any continued fraction may be reduced to a simple fraction; but by inspecting the work, we can arrive at a more simple method. The denominator of the fraction $\frac{3}{4} = \frac{1}{1\frac{1}{3}}$, is formed by multiplying the integral part of the denominator by 3, the denominator of the fractional part, and adding the numerator, which is unity. Again the denominator of the fraction $\frac{4}{11} = \frac{1}{2+\frac{3}{4}}$ is formed in like manner, by taking the product of the two denominators and adding 3 to the result. So also the denominator of $\frac{11}{37} = \frac{1}{3+\frac{4}{11}}$, is equal to $3 \times 11 + 4$. Hence it appears, that the denominator of the required simple fraction is found, by beginning at the last denominator of the continued fraction, and multiplying successively by all the denominators; adding unity to the first product, and to each of the others, always the last previous result. In the reductions, the same number becomes alternately the denominator of a complex fraction, and the numerator of the simple one; and in the last reduction we have $\frac{1}{3+\frac{4}{11}} = \frac{1}{\frac{37}{11}} = \frac{11}{37}$. Hence it appears, that the required numerator is the number obtained previous to the last multiplication. For reducing a continued fraction to a simple one, we have, therefore, the following

RULE.

Beginning at the last denominator, multiply successively by all the denominators, adding unity to the first product, and to each of the others, always the last previous result. This will give the denominator of the required fraction, and the product immediately preceding it, the numerator.

EXAMPLES.

1. Reduce $\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}}}$ to a simple fraction.

$2 \times 2 + 1 = 5$, $5 \times 2 + 2 = 12$, $12 \times 2 + 5 = 29$: 29 is the denominator, and 12, the result immediately preceding it, the numerator.

2. **Reduce** $$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{7+\frac{1}{2}}}}}}}$$ to a simple fraction.

$2\times7+1=15$, $15\times1+2=17$, $17\times1+15=32$, $32\times2+17=81$, $81\times1+32=\overset{\text{Num.}}{113}$, $113\times1+81=\overset{\text{Denom.}}{194}$. $\frac{113}{194}$ is, therefore, the simple fraction.

3. **Reduce** $$\cfrac{1}{1+\cfrac{1}{8+\frac{1}{3}}}$$ to a simple fraction.

Ans. $\frac{25}{28}$.

4. **Reduce** $$\cfrac{1}{3+\cfrac{1}{1+\frac{1}{11}}}$$ to a simple fraction.

Ans. $\frac{12}{47}$.

5. **Reduce** $$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\frac{1}{6}}}}}$$ to a simple fraction.

Ans. $\frac{421}{972}$.

6. Reduce $\cfrac{1}{6+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{3+\frac{1}{2}}}}}$ to a simple fraction.

Ans. $\frac{157}{972}$.

7. Reduce $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{7}}}}$ to a simple fraction.

Ans. $\frac{15}{23}$.

DECIMAL FRACTIONS.

(43.) A decimal fraction is one which has for its denominator an unit, with as many cyphers annexed to it as the numerator has places of figures. It is customary to write the numerator only, since by the number of places *it* contains, the value of the denominator is known. Thus, $\frac{4}{10}$ is written ,4, $\frac{24}{100}$,24 and $\frac{361}{1000}$,361. When the numerator does not contain as many figures as the denominator has cyphers, the deficiency is supplied by writing ciphers at the left hand; thus, $\frac{5}{100}$ is written ,05 $\frac{32}{1000}$,032, and $\frac{11}{10000}$,0011.

A mixed number, made up of a whole number and a decimal fraction, is written with a point called the *decimal point*, between the whole number and the fractional part; thus, 4,67 is the same as $4\frac{67}{100}$, or $\frac{467}{100}$.

Ciphers on the right hand of a decimal do not alter its value, for $\frac{4}{10}$, $\frac{40}{100}$ and $\frac{400}{1000}$, are the same, since they can each be reduced to $\frac{2}{5}$. But ciphers on the left hand do alter the value of the decimal; while ,3 represents $\frac{3}{10}$, ,03 is but $\frac{3}{100}$, and ,003 is but $\frac{3}{1000}$.

From the method of notation adopted in decimal fractions, it is evident that decimals, like whole numbers, decrease from left to right in a tenfold proportion, each place, as we recede from the left, having but one tenth the value of that which precedes it. Thus, if we commence at the left in whole numbers, we find each place representing a unit of a lower order, till we arrive at the lowest, namely, units. Continuing the same law of decrease, the first figure to the right of units becomes 10th parts, the second, 100th parts, the third, 1000th parts, and so on.

We have then only to apply the system of numeration for whole numbers, counting from the decimal point both ways;—towards the left, units, tens, hundreds, &c.; and towards the right, tenth parts, hundredth parts, thousandth parts, &c., as is represented in the following table:—

Millions.	Hundreds of thousands.	Tens of thousands.	Thousands.	Hundreds.	Tens.	Units.	10th parts.	100th parts.	1000th parts.	10,000th parts.	100,000th parts.	1000,000th parts.
2	3	6	7	4	3	1	3	4	7	6	3	2

The expression in the table is read, 2 millions, 367 thousands, 431, and 347 thousands, 632 millionths.

(44.) Decimal fractions result from the division of one number by another in the following manner:—

Let it be required to divide 63 by 8.

```
8)63(7,875
  56
  ——
   70
   64
   ——
    60
    56
    ——
     40
     40
```

I find 8 contained 7 times in 63, with a remainder of 7; I now conceive that each of the units in 7 is divided into 10 parts, which gives me 70 tenth parts of a unit for a new dividend; this is divisible by 8; and since this second dividend is tenth parts, the quotient will be tenth parts; and hence the 8 in the quotient is written with the decimal point before it. I now find a remainder of 6 tenth parts of a unit; each of these parts are also subdivided into ten parts, or, which is the same thing, *this* remainder is multiplied by 10, which gives 60 hundredth parts; the quotient of which, by the divisor, is 7 hundredths, which is put in the place of hundredths in the quotient. The next remainder is likewise multiplied by 10, which gives thousandth parts for a new dividend, and a figure in the place of thousandths, in the quotient.

Hence it appears, that a division may be executed in decimals, when it cannot be in whole numbers. In the preceding example, the quotient of 63 by 8 is 7,875. Expressed by a whole number and vulgar fraction, it would have been $7\frac{7}{8}$. By inspecting the work, it will be seen, that the vulgar fraction $\frac{7}{8}$, is reduced to a decimal by adding cyphers to the numerator (for this is multiplying by 10), and dividing by the denominator. The following rule will now require no explanation.

To reduce a vulgar fraction to a decimal.

RULE.

(45.) *Annex ciphers to the numerator, and divide by the denominator, and point off as many places for decimals in the quotient, as there are ciphers annexed to the numerator.*

EXAMPLES.

1. Reduce $\frac{1}{8}$ to a decimal.

```
8)10(0,125
  8
 ---
  20
  16
 ---
   40
   40
```

2. Reduce $\frac{3}{4}$ to a decimal fraction. Ans. 0,75.

3. Reduce $\frac{1}{2}$ to a decimal fraction. Ans. 0,5

4. Reduce $\frac{4}{7}$ to a decimal fraction.
Ans. 0,57142857142, &c.

5. Reduce $\frac{13}{4}$ to a decimal fraction.
Ans. 3,25.

6. Reduce $\frac{11}{12}$ to a decimal fraction.
Ans. 0,916666, &c.

7. Reduce $\frac{13}{91}$ to a decimal fraction.
Ans. 0,14285714, &c.

8. Reduce $\frac{1}{16}$ to a decimal fraction.
Ans. 0,0625.

9. Reduce $\frac{1}{999}$ to a decimal fraction.
Ans. 0,001001001, &c.

10. Reduce $\frac{5}{716}$ to a decimal fraction.
Ans. 0,0069832, &c.

To reduce a decimal to a vulgar fraction.

RULE.

(46.) *Write down the decimal with its denominator, and then reduce it to its lowest terms.*

EXAMPLES.

1. Reduce ,0625 to a vulgar fraction.
Ans. $\frac{625}{10000} = \frac{125}{2000} = \frac{5}{80} = \frac{1}{16}$.

2. Reduce ,532 to a vulgar fraction.
Ans. $\frac{133}{250}$.

3. Reduce ,0078125 to a vulgar fraction.
Ans. $\frac{1}{128}$.

CIRCULATING DECIMALS.

(47.) Circulating decimals are those, in which the same figures are repeated at regular intervals in the same order; $,32\dot{4}324$, &c., and $,5\dot{4}54$, &c. are circulating decimals. When a single figure is repeated as, ,3333, the decimal is sometimes called a repeating decimal. Circulating decimals result from divisions in which the divisor and dividend are prime to each other, and the divisor is such a number, that it is not contained without remainder, in any number expressed by 1, and any number of ciphers, as 10, 100, 1000, &c. Take, for example, the fraction $\frac{6}{11}$. If we attempt to reduce this fraction to a decimal by the preceding rules, we shall find the same numbers recurring in the remainders; and hence the same numbers will be repeated in the quotient.

EXAMPLE.

$$\begin{array}{l}
11)60(0,5454 \\
55 \\
\hline
\,50 \\
\,44 \\
\hline
\;\;60 \\
\;\;55 \\
\hline
\;\;\;50 \\
\;\;\;44 \\
\hline
\;\;\;\;60
\end{array}$$

The numbers repeated are called periods. In this example two figures form the period.

It is evident, that the true value of this fraction cannot be expressed in a decimal; and it is sometimes convenient, when such a decimal fraction occurs, to change it into its equivalent vulgar fraction. This may be done by the following

RULE.

Write under the figures which constitute one period, as many 9s as there are places of figures in the period. This will form the fraction from which the circulating decimal was derived.

For another example, reduce $32\dot{4}32\dot{4}324$ to a vulgar fraction. Ans. $\frac{324}{999} = \frac{36}{111} = \frac{12}{37}$.

Proof.

$$\begin{array}{r}
37)120(0,324324 \\
111 \\ \hline
90 \\
74 \\ \hline
180 \\
148 \\ \hline
120 \\
111 \\ \hline
90 \\
74 \\ \hline
160 \\
148 \\ \hline
12
\end{array}$$

The correctness of the foregoing rule will appear from the following considerations.

The fractions which have for their denominators any number of 9s, can have no significant figures in their periods, except 1.

$\frac{1}{9}$ gives 0,11111, &c.
$\frac{1}{99}$ " 0,010101.
$\frac{1}{999}$ " 0,001001, &c.

And so of any other; for, each partial division of the numbers 10, 100, 1000, 10000, &c. must always give 1 for a remainder. Now, a circulating decimal, with a period of one figure, must always be some multiple of $\frac{1}{9}$, since 0,33333 is the same as 0,11111 × 3, and 0,66666 is the same as 0,11111 × 6. And, since 0,11111 = $\frac{1}{9}$, 0,11111 × 3, or, 0,33333 = $\frac{3}{9}$ or $\frac{1}{3}$. So also, 0,66666 = $\frac{1}{9}$ × 6, or, $\frac{6}{9} = \frac{2}{3}$. Again, 0,545454 = 0,010101 × 54 = $\frac{1}{99}$ × 54 = $\frac{54}{99} = \frac{6}{11}$. For a period of three figures, we have, 0,324324324 = 0,001001001 × 324 = $\frac{1}{999}$ × 324 = $\frac{324}{999} = \frac{12}{37}$.

EXAMPLES.

1. Reduce $0{,}41\dot{4}14\dot{1}$ to the form of a vulgar fraction. Ans. $\frac{41}{99}$.

2. Reduce $0{,}57142\dot{8}571428$ to the form of a vulgar fraction. Ans. $\frac{4}{7}$.

3. Reduce $0{,}86\dot{4}864$ to the form of a vulgar fraction. Ans. $\frac{32}{37}$.

NOTE.—If the period of the fraction does not begin with the first decimal, we can change the place of the point, and put it before the first figure of the period; and then find the value of the fraction, as if the figures to the left were units. The result must then be divided by 10, 100, 1000, &c., according to the number of places the point was moved to the right.

EXAMPLES.

1. Reduce $0{,}8\dot{3}\dot{3}3$ to the form of a vulgar fraction.

Moving the point one place to the right, we have, for the value of the circulating decimal, $\frac{1}{3}$; and considering 8 as a whole number, the expression for the fraction would be $8\frac{1}{3}$; but 8 is in the place of 10ths. We, therefore, divide by 10, which gives $\frac{8}{10}+\frac{1}{30}$, or $\frac{24}{30}+\frac{1}{30}=\frac{25}{30}=\frac{5}{6}$, which will reproduce the decimal.

For another example, reduce $0{,}621\dot{4}\dot{5}45$ to the form of a vulgar fraction.

$621{,}45=621\frac{45}{99}$, or $621\times\frac{5}{11}$, and dividing by 1000, $\frac{621}{1000}+\frac{5}{11000}=\frac{6831}{11000}+\frac{5}{11000}=\frac{6836}{11000}=\frac{1709}{2750}$.

2. Reduce $0{,}9\dot{1}23\dot{1}23$ to the form of a vulgar fraction. Ans. $\frac{1519}{1665}$.

3. Reduce $0{,}99\dot{9}\dot{0}90$ to the form of a vulgar fraction. Ans. $\frac{1099}{1100}$.

4. Reduce $0{,}2\dot{4}1\dot{8}418$ to the form of a vulgar fraction. Ans. $\frac{1208}{4995}$.

ADDITION OF DECIMALS.

RULE.

(48.) *Set down the numbers according to their value, so that tenths may be under tenths, hundredths under hundredths, &c., and proceed as in addition of whole numbers; and point off, in the sum, as many places of figures for decimals, as were contained in the number having the greatest number of decimal places.*

Since in decimals, as in whole numbers, the value of the places increases from right to left in a tenfold proportion (44), it is evident, that the same rule will apply to both; and it is only necessary in the former to have regard to the position of the decimal points, that it may be evident where the whole numbers terminate, and the decimals begin.

EXAMPLES.

1. Find the sum of 29,0146, 3146,5, 2109, 62417, and 14,16.

```
  29,0146
3146,5
2109,
   0,62417
  14,16
──────────
5299,29877 the sum.
```

2. What is the sum of 276, 39,213, 72014,9, 417, and 5032,17?

3. What is the sum of 312,09, 71,0003, 92, 1134, 0,76, and 0,003?

4. What is the sum of 1 tenth, 2 hundredths, 9 ten thousandths, and 1? Ans. 1,1209.

5. What is the sum of 416 millionths, 312 thousandths, 13 hundredths, 8 tenths, and 18? Ans. 19,242416.

SUBTRACTION OF DECIMALS.

Rule.

(49.) *Set down the numbers according to their values, and subtract as in whole numbers, placing the decimal point in the remainder, directly under the place it occupied in the subtrahend.*

Examples.

1. Find the difference between 91,73 and 2,138.

$$\begin{array}{r} 91,73 \\ 2,138 \\ \hline 89,592 \end{array}$$ the difference.

Note.—If the minuend does not contain as many decimal places as the subtrahend, we may conceive the places to be filled out with 0's, since 0's at the right of a decimal do not affect its value.

2. Find the difference between 387,92461 and 33,4672.
Ans. 354,45741.

3. Find the difference between 28 and 0,9289.
Ans. 27,0711.

4. Find the difference between 1 tenth and 99 thousandths
Ans. 0,001.

5. Find the difference between 1 and 1 millionth.
Ans. 0,999999.

MULTIPLICATION OF DECIMALS.

Rule.

(50.) *Place the factors, and multiply, as in whole numbers; and point off, in the product, as many figures for decimals as there are decimal places in both the factors. But if there be not so many figures in the product, supply the defect by prefixing ciphers.*

Demonstration.—This rule follows directly from the system of notation in decimals. A denominator, expressed by 1, and as many 0's as there are places of figures in the decimal, being always understood (43), it is plain that if we multiply two decimal fractions together, the product will have, for its denominator, 1, and as many 0's as there are in both denominators; that is, as many 0's as there are places in both numerators; and hence, in the resulting decimal, as many figures must be taken for decimals, as there were in both the factors.

Let it be required to multiply 0,361 by 0,12. These decimals are equivalent to $\frac{361}{1000}$ and $\frac{12}{100}$; multiplying these by the rule for vulgar fractions (22), we have $\frac{361}{1000}\times\frac{12}{100}=\frac{43332}{100000}=$, by the decimal notation, to 0,04332.

The same would be true of a number consisting of a whole number and a decimal, since it may be reduced to an improper fraction, which must always have for its denominator 1, and as many 0's as there are places of figures in the decimal part; thus, 9,113 is the same as $9\frac{113}{1000}$ or, $\frac{9113}{1000}$.

EXAMPLES.

1. Multiply 0,321096
by ,02465

1605480
1926576
1284384
642192

0,00791501640 Product.

2. Multiply 79,347
by 23,15

1836,88305 Product.

3. Multiply 0,385746
by 0,00464

0,00178986144 Product.

4. Multiply 9 by 0,00007. Ans. 0,00063.

5. Multiply 0,000031 by 5,000002.
Ans. 0,000155000062.

NOTE.—To multiply by 10, 100, 1000, &c., remove the decimal point as many places to the right as there are 0's in the multiplier. Thus, 23467×10=234,67, 6,7234×100=672,34, and 5,71 × 1000=5710.

DIVISION OF DECIMALS.

RULE.

(51.) *Divide as in whole numbers, and point off, in the quotient, as many figures for decimals, as the decimal places in the dividend exceed those in the divisor.*

The reason for this rule is evident, for the quotient multiplied by the divisor, must produce the dividend; and hence the number of decimal places in the dividend, must be equal to those in the divisor and quotient taken together; and, consequently, the quotient must contain as many decimal places as the places in the dividend exceed those in the divisor.

EXAMPLES.

```
1.—1,78)0,48520998(0,272589
        356
        ----
        1292
        1246
        ----
          460
          356
          ----
          1049
           890
          ----
           1599
           1424
           ----
            1758
            1602
            ----
             156
```

2. Divide 123,70536 by 54,25. Quo. 2,2802.

3. Divide 12 by 0,7854. Quo. 15,278.

4. Divide 4195,68 by 100. Quo. 41,9568.

5. Divide 0,8297592 by 0,153. Quo. 5,4232.
6. Divide 27 by 0,2639. Quo. 102,3114.

NOTE.—To divide by 10, 100, 1000, &c., remove the decimal point as many places to the left as there are 0's in the divisor. Thus,

167,354 ÷ 10 = 16,7354, 9217,3719 ÷ 100 = 92,173719.

CHAPTER III.

INVOLUTION AND EVOLUTION.

(52. Involution is the multiplication of a number one or more times by itself.

The product of a number thus multiplied by itself, is called the power of that number.

The number of times the given number is a factor in a power, is called the degree of that power.

Thus, 2 = 2, is the 1st power of 2.
2 × 2 = 4, is the 2d power of 2.
2 × 2 × 2 = 8, is the 3d power of 2.
2 × 2 × 2 × 2 = 16, is the 4th power of 2.
2 × 2 × 2 × 2 × 2 = 32, is the 5th power of 2.

The number itself, with reference to its powers, is called the root; thus, 2 is the 2d or square root of 4, the 3d or cube root of 8, the 4th root of 16, &c.

Table of the first Nine Powers of Numbers.

1st.	2d.	3d.	4th.	5th.	6th.	7th.	8th.	9th.
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489
10	100	1000	10000	100000	1000000	10000000	100000000	1000000000

The degree of the power to which a number is to be raised, is indicated by writing to the right of the given number, and a little above, a small figure expressing the degree. Thus, $2^3 = 8$ is the 3d power of 2. This figure is called the index or exponent of the power.

$4^2 = 16$ is the 2d power of 4.
$2^6 = 64$ is the 6th power of 2.
$13^4 = 28561$ is the 4th power of 13.

(53.) When two or more powers are multiplied together, their product is that power whose exponent is the sum of the exponents of the powers multiplied; or, the multiplication of the powers, answers to the addition of the exponents, as will appear from the following example of the powers of 2:—

	1st.	2d.	3d.	4th.	5th.	6th.	7th.	8th.	9th.	10th.
	2,	4,	8,	16,	32,	64,	128,	256,	512,	1024,
or,	2^1,	2^2,	2^3,	2^4,	2^5,	2^6,	2^7,	2^8,	2^9,	2^{10}.

By multiplying the 2d and 4th, we have $4 \times 16 = 64$, and adding the exponents from the second line, $2^2 \times 2^4 = 2^6 = 64$. We have also,

$8 \times 32 = 256$, and $2^3 \times 2^5 = 2^8 = 256$,
$4 \times 256 = 1024$, and $2^2 \times 2^8 = 2^{10} = 1024$,
$16 \times 64 = 1024$, and $2^4 \times 2^6 = 2^{10} = 1024$,
$8 \times 128 = 1024$, and $2^3 \times 2^7 = 2^{10} = 1024$.

To involve a number to a given power.

RULE.

(54.) *Multiply the given number by itself a number of times less 1, than the exponent of the required power; or take two or more powers of the given number the sum of whose exponents is equal to the exponent of the required power, and multiply them together successively.*

EXAMPLES.

1. Involve 67 to the 2d power.
$67 \times 67 = 4489$, the answer.

2. What is the 3d power of 93?
$93 \times 93 \times 93 = 804357$, the answer.

3. What is the 8th power of 4?

$4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 65536$;

or, $4^4 \times 4^4 = 4^8$; that is, $256 \times 256 = 65536$;

or, $4^2 \times 4^3 \times 4^3 = 4^8$; that is, $16 \times 64 \times 64 = 65536$, the ans.

4. What is the square of 4,16?
Ans. 17,3056.

5. What is the 3d power of 3,5?
Ans. 42,875.

6. What is the 4th power of 0,05?
Ans. 0,00000625.

7. What is the 2d power of 899?
Ans. 808201.

8. What is the 6th power of 26?
Ans. 308915776.

9. What is the 2d power of $\frac{2}{3}$?
$\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, the answer.

10. What is the 4th power of $\frac{1}{2}$?
$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$, the answer.

11. What is the 3d power of $\frac{5}{9}$?
Ans. $\frac{125}{729}$.

12. What is the 5th power of $\frac{3}{4}$?
Ans. $\frac{243}{1024}$.

13. What is the 3d power of $3\frac{1}{5}$?
$3\frac{1}{5} = \frac{16}{5}$ and $(\frac{16}{5})^3 = \frac{4096}{125}$, or, $32\frac{96}{125}$, the ans.

14. What is the 2d power of $4\frac{1}{2}$?
Ans. $\frac{81}{4}$, or, $20\frac{1}{4}$.

15. What is the 4th power of $2\frac{2}{3}$?
Ans. $\frac{4096}{81}$, or, $50\frac{46}{81}$.

EVOLUTION.

(55.) Evolution is the reverse of involution. It is the extracting or finding the roots of any given powers.

The *root* of any number, is such a number as multiplied into itself a certain number of times, will produce that number; thus,

5 is the square or 2d root of 25, since 5^2, or, $5 \times 5 = 25$;

4 is the cube or 3d root of 64, since 4^3, or, $4 \times 4 \times 4 = 64$.

5. Divide 0,8297592 by 0,153. Quo. 5,4232.

6. Divide 27 by 0,2639. Quo. 102,3114.

NOTE.—To divide by 10, 100, 1000, &c., remove the decimal point as many places to the left as there are 0's in the divisor. Thus,

$167,354 \div 10 = 16,7354$, $9217,3719 \div 100 = 92,173719$.

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INVOLUTION AND EVOLUTION.

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$2 \times 2 \times 2 \times 2 = 16$, is the 4th power of 2.
$2 \times 2 \times 2 \times 2 \times 2 = 32$, is the 5th power of 2.

The number itself, with reference to its powers, is called the root; thus, 2 is the 2d or square root of 4, the 3d or cube root of 8, the 4th root of 16, &c.

EXAMPLES.

1. What is the square root of 29506624 ?

29506624(5432, the root.
25

104(450
416

1083)3466
3249

10862)21724
21724

NOTE I.—There are many numbers the square root of which cannot be extracted exactly; but we can arrive very nearly at the root by continuing the operation in decimals; and this is done by adding periods of two ciphers after the whole numbers are all brought down.

2. For example, let it be required to find the square root of 312.

312(17,6635, the root.
1

27)212
189

346)2300
2076

3526)22400
21156

35323)124400
105969

353265)1843100
1766325

76775

3. Find the square root of 2.

2)1,41421 &c., the root.
1

24)100
96

281)400
281

2824)11900
11296

28282)60400
56564

282841)383600
282841

100759

NOTE II.—If the number consist of a whole number and a decimal, it is necessary to divide the decimal part into periods, beginning at the decimal point, and counting two figures to the right for each period; and if the right hand period of the decimal should contain but one figure, a cipher must be annexed to it.

4. Let it be required to find the square root of 220,758

220,7580(14,85
1

24)120
96

288)2475
2304

2965)17180
14825

2355

5. What is the square root of 2025? Ans. 45.

6. What is the square root of 17,3056? Ans. 4,16

7. What is the square root of 0,000729? Ans. 0,027.
8. What is the square root of 3? Ans. 1,732050 &c.
9. What is the square root of 343396? Ans. 586.
10. What is the square root of 300304? Ans. 548.
11. What is the square root of 220900? Ans. 470.
12. What is the square root of 10? Ans. 3,162277 &c.
13. What is the square root of 1883,56? Ans. 43,4.

Note III.—The square root of a vulgar fraction is found, by extracting the roots of the numerator and denominator. But, if either the numerator or denominator should not be a perfect square, the fraction should be reduced to a decimal, and the approximate root extracted.

Examples.

1. What is the square root of $\frac{49}{64}$? Ans. $\frac{7}{8}$
2. What is the square root of $\frac{27}{147}$?
   $\frac{27}{147} = \frac{9}{49}$, hence $\frac{3}{7}$ is the square root.
3. What is the square root of $\frac{5}{12}$?
   Ans. $\frac{5}{12} = 0{,}41666666$, and $\sqrt{0{,}41666666} = 0{,}6454$.
4. What is the square root of $17\frac{3}{8}$? Ans. 4,168333.

EXTRACTION OF THE CUBE ROOT.

Rule.

(57.) *Divide the given number into periods of three figures each, beginning at the right; and if there be decimals, point off the decimals in a contrary direction from the decimal point; observing, that if the last period of decimals does not contain three figures, the deficiency is to be supplied by annexing ciphers.*

Find the greatest cube in the left hand period, and place its root at the right, after the manner of a quotient in division. Subtract its cube from the aforementioned period, and to the remainder bring down the next period for a dividend.

Multiply the square of the root already found by 300, *for a divisor. Seek how often the divisor is contained in the dividend, and place the result in the root.*

Complete the divisor by adding to it thirty times the product of the root before found by this last figure, and also the square of the last figure; then multiply the divisor, so increased, by the same figure, and subtract the product from the dividend.

To the remainder bring down the next period, and proceed as before; and continue the operation in the same manner, through all the periods of the given number.

NOTE I.—If the last mentioned product should be greater than the dividend, the last figure in the root must be diminished.

EXAMPLES.

1. Find the cube root of 178834180,104.

		178834180,104	563,4
		125	
	divisor.		
$5^2 = 25 \times 300 =$	7500	53834	1st dividend.
$5 \times 6 \times 30 + 36$	936		
Divisor completed,	8436	50616	1st subtrahend.
	2d divisor.		
$(56)^2 = 3136 \times 300 =$	940800	3218180	2d dividend.
$56 \times 3 \times 30 + 9 =$	5049		
2d divisor completed,	945849	2837547	2d subtrahend.
	3d divisor.		
$(563)^2 = 316969 \times 300 =$	95090700	380633104	3d dividend.
$563 \times 4 \times 30 + 16 =$	67576		
3d divisor completed,	95158276	380633104	3d subtrah'd.
		0	

2. Find the cube root of 970299. Ans. 99.
3. Find the cube root of 7645373. Ans. 197.
4. Find the cube root of 14,348907. Ans. 2,43.
5. Find the cube root of 179406144. Ans. 564.
6. Find the cube root of 533411,731. Ans. 81,1.

NOTE II.—To find the cube root of a fraction, apply the same principles as in the square root, thus :—

$$\sqrt[3]{\frac{8}{27}}=\frac{\sqrt[3]{8}}{\sqrt[3]{27}}=\frac{2}{3};$$

$$\sqrt[3]{\tfrac{7}{8}} = \sqrt[3]{0{,}875} = 0{,}9564, \text{ \&c.}$$

$$\sqrt[3]{15\tfrac{5}{8}} = \sqrt[3]{\tfrac{125}{8}} = \tfrac{5}{2};$$

$$\text{or, } \sqrt[3]{15\tfrac{5}{8}} = \sqrt{15{,}625} = 2{,}5.$$

Find the cube root of $\frac{11}{49}$.

Ans. $\frac{2{,}223 \text{ \&c.}}{3{,}659 \text{ \&c.}}$, or 0,6077 &c.

Find the cube root of $\frac{23}{50}$.

Ans. 0,77194, or $\frac{2{,}843 \text{ \&c.}}{3{,}684 \text{ \&c.}}$

CHAPTER IV.

RATIOS AND PROPORTIONS.

(58.) Any relation which subsists between two numbers is called a ratio.

An *arithmetical ratio* is the difference of two numbers, and is expressed by writing the less number after the greater, with the sign of subtraction between them. Thus, the arithmetical ratio of 7 to 3, is written 7 — 3; that of 9 to 4, is written 9 — 4.

A *geometrical ratio* is the quotient of one number divided by another, or the number of times one number is contained in another. The particular value of this quotient is called the index of the ratio. The geometrical ratio of 12 to 4, is written 12 : 4, or $\frac{12}{4}$; and 3, the quotient of 12 divided by 4, is the index of the ratio.

Ratio can only exist between numbers expressing quantities of the same kind; as it would be absurd to enquire, how much more than a rod is two pounds? or how many times five yards of cloth is contained in forty dollars?

Two ratios may be equal to each other, and such ratios, when written together, with the sign of equality between them, constitute a *proportion.* Thus, the arithmetical ratio 9 — 4, is equal to the ratio 12 — 7, and from these two ratios, the proportion 9 — 4 = 12 — 7, may be formed.

The *geometrical ratio,* 12 : 4, or $\frac{12}{4}$, is equal to the ratio 18 : 6, or $\frac{18}{6}$; and these two ratios form the *proportion,* 12 : 4 = 18 : 6, or $\frac{12}{4} = \frac{18}{6}$.

The first term of a ratio is called the *antecedent,* and the second term the *consequent.* The first and last terms of a proportion are called the *extreme terms,* the second and third the *mean terms;* or simply, the *extremes* and *means.*

ARITHMETICAL PROPORTIONS.

(59.) An arithmetical ratio being the difference of two numbers, and an arithmetical proportion expressing an equality between two ratios; it follows, that an arithmetical proportion shows that the difference of two given numbers is equal to the difference of two others. Thus, 12 — 7 = 9 — 4, shows that the difference of 12 and 7 is equal to the difference of 9 and 4. Another propriety of arithmetical proportions, and one on which most of the calculations relating to them are based, is the following:—

In every arithmetical proportion, the sum of the means is equal to the sum of the extremes.

To prove this, take the proportion 12 — 7 = 9 — 4. Since equals added to equals make equal sums, the quantities on each side of the sign of equality will still be equal, if 7 be added to each; and we shall have 12 — 7 + 7 = 9 — 4 + 7: adding also 4 to each, we shall have

$$12 - 7 + 7 + 4 = 9 - 4 + 7 - 4.$$

But since a quantity both added to, and subtracted from, another, can neither increase nor diminish it, we may expunge 7 from the first, and 4 from the second, and there remains 12 + 4 = 9 + 7; that is, the sum of the extremes equal to the sum of the means. We have now proved this property to belong to the particular proportion $12-7=9-4$; but to show that it is general and independent of the particular numbers used, let a, b, c, and d, represent any numbers whatever which have this property, namely, that the difference between the numbers represented by a and b, is equal to the difference of those represented by c and d. They will evidently form the arithmetical proportion,

$$a-b=c-d.$$

If we now add d to each of these equal quantities, we shall have

$$a-b+d=c-d+d;$$

and adding b to each,

$$a-b+d+b=c-d+d+b;$$

and as b is both added to, and subtracted from, the first, and d added to, and subtracted from, the second, b may be omitted in the first, and d in the second, and we have

$$a+d=c+b;$$

or the sum of the extremes, equal to the sum of the means.

The letters of the alphabet are here introduced to accustom the pupil to their use, preparatory to his entering upon the study of Algebra. He will easily perceive, that they are here subjected to the same arithmetical operations as the numbers in the preceding proportion, and will see how, by means of *signs*, we can apply the rules of arithmetic to letters as the representatives of numbers without giving to the letters any particular numerical value. Thus, in the proportion $a-b=c-d$, any numbers whatever which have such a relation to each other, that the second shall be as much less than the first, as the fourth is less than the third, may be put in the place of the letters; and all that we have proved of the letters, will be true of the numbers.

Suppose $a=21$, $b=12$, $c=19$, and $d=10$; then

$$21-12=19-10;$$

and since we had $a+d=c+b$, we shall have

$$21+10=12+19.$$

Again, let $a=11$, $b=6$, $c=18$ and $d=13$, then

$$11-6=18-13,$$

$$\text{and}\quad 11+13=6+18.$$

(60.) From the property which has just been demonstrated, it follows, that if three terms of an arithmetical proportion be given, the fourth can always be found by the following rule.

RULE.

If the unknown term be one of the extremes, add together the mean terms, and subtract the other extreme; if it be a mean term, add the extremes and subtract the given mean.

Let there be the arithmetical proportion

$$19 - 7 = 23 -,$$

and we shall have for the fourth term,

$$7 + 23 - 19 = 30 - 19 = 11.$$

In the proportion $13 - 5 = . - 12$, the third term is found by adding the extremes, and subtracting the known mean terms; thus,

$$13 + 12 - 5 = 25 - 5 = 20, \text{ the third term,}$$

and we have the complete proportion,

$$13 - 5 = 20 - 12.$$

When the consequent of the first ratio is the same as the antecedent of the second, the proportion is called a continued arithmetical proportion. $18 - 15 = 15 - 12$ is a continued arithmetical proportion; the number 15 being the consequent of the first ratio, and the antecedent of the second.

To such a proportion another ratio, or any number of ratios may be added, all having this same property, and this series of ratios is called an arithmetical series, or progression: thus,

$$18 - 15 = 15 - 12 = 12 - 9 = 9 - 6 = 6 - 3 = 3 - 0,$$

is an arithmetical progression.

It is usual, in arithmetical progression, to omit one of the mean terms of each proportion, and also the sign of equality. The progression above would be written,

$$18 - 15 - 12 - 9 - 6 - 3 - 0.$$

Arithmetical progressions are usually written without the sign — between the terms. When the numbers increase from left to right, the series is called an increasing arithmetical progression; when they increase in a contrary order, a decreasing arithmetical progression.

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19,$$

is an increasing progression;

$$21, 19, 17, 15, 13, 11, 9, 7, 5, 3, 1,$$

is a decreasing progression.

The numbers that enter into an arithmetical progression, are called the *terms* of the progression. The first and last terms are the *extremes*, and the number by which each term is greater or less than that which preceded it, is called the *common difference.*

We proceed now to give certain properties of arithmetical progressions, without demonstrations.*

(61.) 1st. *In every arithmetical progression, the sum of the extremes is equal to the sum of any two terms equally distant from them; or, equal to double the middle term, when there is an odd number of terms.*

Thus, in the progression, 2, 4, 6, 8, 10, 12, 14, we have,

$$2+14=4+12=6+10=8+8, \text{ or } 2\times 8.$$

(62.) 2d. *In an increasing arithmetical progression, the last term is equal to the first term,* plus *the common difference multiplied by the number of terms less* 1*; and in a decreasing arithmetical progression, the last term is equal to the first,* minus *the common difference multiplied by the number of terms less* 1.

Thus, in the progression 3, 7, 11, &c., the 12th term is

$$3+(4\times 11)=3+44=47.$$

In the progression 82, 79, 76, &c., the 25th term is

$$82-(3\times 24)=82-72=10.$$

EXAMPLES.

1. The first term of an increasing arithmetical progression is 5, the common difference 6, what is the 10th term? Ans. 59.

2. The 10th term of an arithmetical progression is 34, the common difference 3, required the first term?

In a question like this, we may consider 34 the first term of a decreasing progression, and find the 10th term, which would correspond to the 1st term in an increasing progression;

whence, $34-(3\times 9)=34-27=7$, the ans.

3. The 1st term of a progression is 2, the common difference $\frac{1}{3}$, what is the 27th term? Ans. $10\frac{2}{3}$.

4. The 12th term of a progression is 3, the common difference $\frac{1}{5}$, what is the 1st term? Ans. $\frac{4}{5}$.

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* The demonstrations will be given in Algebra.

(63.) 3d. *In every arithmetical progression, the common difference is equal to the difference of the extreme terms, divided by the number of terms less* 1.

Thus, if the extremes of a progression are 3 and 58, and the number of terms 12, we have for the common difference,

$$\frac{58-3}{11}=\frac{55}{11}=5.$$

Examples.

1. If the extremes are 1 and 21, and the number of terms 11, what is the common difference? Ans. 2.

2. The extremes of an arithmetical progression are 5 and 23, the number of terms 25, what is the common difference? Ans. $\frac{3}{4}$.

(64.) 4th. *In every arithmetical progression, the sum of all the terms is equal to the sum of the extremes, multiplied by the number of terms and divided by* 2.

Thus, if the extremes of a progression be 7 and 82, and the number of terms 16,

$$7+82=89, \text{ and } \frac{89\times 16}{2}=\frac{1424}{2}=712.$$

Examples.

1. What is the sum of the first 7 terms of the progression 6, 10, 14, &c.?

$$6+4\times 6=6+24=30, \text{ the 7th term.}$$

$$\text{Then, } \frac{30+6}{2}\times 4=\frac{144}{2}=72, \text{ the sum.}$$

2. The first term of a progression is 3, the common difference $\frac{3}{8}$; required the sum of the first 21 terms. Ans. $141\frac{3}{4}$.

3. Find the sum of the progression 2, 5, 8, 11, &c. to 31 terms. Ans. 1457.

GEOMETRICAL PROPORTIONS.

(65.) *In every geometrical proportion, the product of the means is equal to the product of the extremes.*

Take the proportion $12 : 4 = 18 : 6$, or $\frac{14}{4} = \frac{18}{6}$. Now, if we reduce the fractions $\frac{12}{4}$ and $\frac{18}{6}$ to a common denominator, we shall not alter their values, and we shall have

$$\frac{12\times 6}{4\times 6} = \frac{18\times 4}{4\times 6}.$$

The numerators of these fractions must be equal, since their denominators are equal, and the fractions themselves equal; but the numerator of the first is the product of the extreme terms of the proportion, and the numerator of the second is the product of the mean terms; *hence, in every geometrical proportion, the product of the means is equal to the product of the extremes.*

The proof is independent of the particular numbers which enter into the proportion, as may be shown by the use of letters, which may stand for any numbers whatever. Let a, b, c and d, represent any numbers whatever, which have such a relation to each other that the first divided by the second, shall be equal to the third divided by the fourth. They will give the proportion

$$a : b = c : d, \text{ or } \frac{a}{b} = \frac{c}{d}.$$

Indicating by means of signs, the same arithmetical operations as were performed upon the proportion above, we have, by reducing the fractions $\frac{a}{b}$ and $\frac{c}{d}$ to a common denomitor,

$$\frac{a\times d}{b\times d} = \frac{c\times b}{b\times d}.$$

In this, as in the former case, the fractions being equal, and the denominators the same, the numerators must also be equal; and hence, the product of the means equal to the product of the extremes.

(66.) The property of geometrical proportions which has just been proved, is the fundamental principle of proportions;

and any changes may be made in a proportion, provided that the equality of the products of the means and extremes be not destroyed.

We may, in the proportion $12 : 4 = 18 : 6$;
transpose the mean terms, as $12 : 18 = 4 : 6$;
transpose the extreme terms, as $6 : 4 = 18 : 12$;
change consequents for antecedents, as $4 : 12 = 6 : 18$;

the product of the means being equal in each, to the product of the extremes.

(67.) We may also multiply by the same number,

the two first terms, as $12 \times 2 : 4 \times 2 = 18 : 6$;
the two last terms, as $12 : 4 = 18 \times 3 : 6 \times 3$;
the antecedents, as $12 \times 5 : 4 = 18 \times 5 : 6$;
the consequents, as $12 : 4 \times 6 = 18 : 6 \times 6$;
all the terms, as $12 \times 3 : 4 \times 3 = 18 \times 3 : 6 \times 3$.

(68.) We can divide by the same number,

the two first terms, as $\frac{12}{2} : \frac{4}{2} = 18 : 6$;
the two last terms, as $12 : 4 = \frac{18}{3} : \frac{6}{3}$;
the antecedents, as $\frac{12}{3} : 4 = \frac{18}{3} : 6$;
the consequents, as $12 : \frac{4}{2} = 18 : \frac{6}{2}$;
all the terms, as $\frac{12}{4} : \frac{6}{4} = \frac{18}{4} : \frac{6}{4}$;

(69.) We may also make the following changes by means of additions and subtractions of the different terms :—

1. By adding the antecedents to the consequents, and comparing the sum with the antecedents, as

$$12 + 4 : 12 = 18 + 6 : 18;$$

and comparing with the consequents,

$$12 + 4 : 4 = 18 + 6 : 6.$$

2. By subtracting the consequents from the antecedents, and comparing with the antecedents, as

$$12-4 : 12 = 18-6 : 18;$$

and comparing with the consequents, as

$$12-4 : 4 = 18-6 : 6.$$

3. By comparing the sum and difference of antecedents and consequents, as

$$12+4 : 12-4 = 18+6 : 18-6.$$

(70.) One proportion may be multiplied or divided by another, term by term, and the result will also be a proportion.

Let the two proportions $\left\{ \begin{array}{l} 12 : 4 = 18 : 6 \\ 10 : 5 = 16 : 8 \end{array} \right\}$ be multiplied term by term.

The result is,

$$12 \times 10 : 4 \times 5 = 18 \times 16 : 6 \times 8,$$

$$\text{or, } 120 : 20 = 288 : 48,$$

in which it will be perceived, that the product of the mean terms is still equal to the product of the extremes.*

(71.) A proportion may also be multiplied by itself, term by term, any number of times; or which is the same thing, each term may be squared, or cubed, or raised to any power whatever, and the resulting numbers will be proportional, as will be seen from the following examples :—

$$4 : 3 = 12 : 9.$$

The square, $16 : 9 = 144 : 81.$

The cube, $64 : 27 = 1728 : 729.$

The fourth power, $256 : 81 = 20736 : 6561.$

(72.) When the mean terms of a proportion are the same, it is called a continued proportion.

$$16 : 8 = 8 : 4,$$

is a continued proportion; and 8 is called a mean proportional between 4 and 16.

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* The same thing may be proved in a different way; for if the proportions be written in a fractional form, we shall see that the multiplication of one proportion by another, is nothing more than the multiplication of equal fractions by equal fractions. The products must also be equal fractions, and consequently, their terms proportional. Thus, $\frac{12}{4} = \frac{18}{6}$ and $\frac{10}{5} = \frac{16}{8}$, being multiplied together, term by term, give $\frac{12}{4} \times \frac{10}{5} = \frac{18}{6} \times \frac{16}{8}$, or $\frac{12 \times 10}{4 \times 5} = \frac{18 \times 16}{6 \times 8}$, which is the same as $12 \times 10 : 4 \times 5 = 18 \times 16 : 6 \times 8$. The same demonstration will, evide[illegible]ly to the squares, cubes, &c. of proportions.

Such a proportion may be continued by the addition of successive ratios, all equal to each other. This forms a geometrical series, or progression.

$$32 : 16 = 16 : 8 = 8 : 4 = 4 : 2 = 2 : 1 = 1 : \tfrac{1}{2} = \tfrac{1}{2} : \tfrac{1}{4}, \text{ \&c.}$$

is a geometrical progression.

It is usual, in writing a geometrical progression, to omit one of the mean terms of each proportion, and connect the terms by the sign :, in the following manner:—

$$32 : 16 : 8 : 4 : 2 : 1 : \tfrac{1}{2} : \tfrac{1}{4} \text{ \&c.}$$
$$1 : 3 : 9 : 27 : 81 : 243 : 729 : 2187 \text{ \&c.}$$

The first of these is called a decreasing progression, and the second an increasing progression.

The first and last terms of a progression are called the extremes. The quotient, arising from the division of any term by the one which precedes it, is called the index of the ratio, or simply the ratio of the progression. In the above examples, $\frac{1}{2}$ is the ratio of the first, since each term is $\frac{1}{2}$ of that which precedes it: and in the second, 3 is the ratio, each term being three times that which precedes it.

We proceed to make known some other properties of geometrical progressions, which cannot be demonstrated without the aid of Algebra.

(73.) 1st. *In every geometrical progression, the product of the two extremes is equal to the product of any two terms equally distant from them, or equal to the square of the middle term, when there is an odd number of terms.*

For example, in the progression

$$2 : 6 : 18 : 54 : 162 : 486 : 1458, \text{ we have}$$
$$2 \times 1458 = 6 \times 486 = 18 \times 162 = 54 \times 54 = 2916.$$

(74.) 2d. *In every geometrical progression, the last term is equal to the first, multiplied by such a power of the ratio as is denoted by an exponent 1 less than the number of terms.*

For example, in the progression,

$$3 : 6 : 12 : 24 : 48 : 96 : 192.$$
$$2^6 = 64 \times 3 = 192, \text{ the seventh term.}$$

EXAMPLES.

1. In a geometrical series, the first term is 3, the ratio 4; what is the sixth term?

$$4^5 = 1024 \times 3 = 3072, \text{ the sixth term.}$$

2. In a geometrical series, the first term is 1, the ratio 3; what is the tenth term? Ans. 19683.

3. In a geometrical series, the first term is 4, the ratio 2; what is the 22d term? Ans. 8388608.

4. If the first term be 64, and the ratio $\frac{1}{4}$, what is the 9th term? Ans. $\frac{1}{1024}$.

(75.) 3d. *The sum of a geometrical progression is equal to the product of the last term by the ratio* minus *the first term, divided by the ratio less* 1.

Hence, the sum of the series;

$$2 : 6 : 18 : 54 : 162 : 486 : 1458,$$

$$\text{is } \frac{(1458\times3)-2}{3-1} = \frac{4374-2}{3-1} = \frac{4372}{3-1} = \frac{4372}{2} = 2186.$$

EXAMPLES.

1. The last term of a geometrical progression is 729, the ratio 3, and the first term 1, what is the sum? Ans. 1093.

2. The first term of a series is 2, the ratio 4, the number of terms 13; what is the sum? Ans. 44739242.

3. The first term of a series is 288, the ratio $\frac{1}{3}$, the number of terms 10; required the sum. Ans. 431 $\frac{2171}{2187}$.

4. The first term of a progression is $\frac{1}{3}$, the ratio $\frac{1}{4}$, and the number of terms 7; what is the sum of the terms? Ans. $\frac{5461}{12288}$.

(76.) 4th. *The ratio of any geometrical progression is equal to the difference between the sum of the terms and the first term, divided by the difference between this sum and the last term.*

The first term of a geometrical series is 2, the last term 1458, and the sum of all the terms 2186; what is the ratio?

$$\frac{2186-2}{2186-1458} = \frac{2184}{728} = 3, \text{ the ratio.}$$

EXAMPLES.

1. The first term of a geometrical progression is 4, the last term 62500, the number of terms 7; what is the ratio?

Ans. 5.

2. The first term of a geometrical progression is $\frac{1}{2}$, the last term $\frac{243}{64}$, and the sum of the terms $\frac{665}{64}$; what is the ratio?

Ans. $\frac{3}{2}$.

RULE OF THREE.

(77.) In any geometrical proportion, if three terms be given, the fourth can always be found; for, if it be one of the extremes, it is equal to the product of the means divided by the other extreme. If the unknown term be a mean, it is equal to the product of the extremes divided by the other mean. This principle is the foundation of what is termed, in arithmetic, the Rule of Three.*

Suppose it be required to find the fourth term of the proportion $12 : 4 = 18 :$ —.

Putting x in the place of the unknown term, the proportion becomes $12 : 4 = 18 : x$; and since the product of the means is equal to the product of the extremes (65), we have $12 \times x = 18 \times 4$, or $12x = 72$; and as 12 times $x = 72$, it is evident that x, the unknown term, is equal to $\frac{72}{12}$, or $\frac{1}{12}$ of 72, that is, equal to 6.

Again, let it be required to find the 3d term of the proportion $6 : 9 = x : 12$: we have as before,

$$9 \times x = 6 \times 12, \text{ or } 9 \times x = 72;$$

---

* We shall now introduce an unknown quantity, denoted by a letter of the alphabet, and perform upon it by means of the signs of arithmetic, all the operations which we should perform upon the number itself were it known. Suppose such a number be represented by the letter x, and it is required to multiply it by 5, we should indicate the multiplication, precisely in the same way, as if it were a number to be multiplied by 5. For the number we should write 5×12, and for the unknown quantity $5 \times x$, or simply $5x$, without the sign. Suppose 10×21 is to be divided by 2, the operation would be performed by dividing 10 by 2, and the result is 5×21. Were it $10 \times x$ to be divided by 2, or simply $10x$, the result would be $5 \times x$, or $5x$. On the same principle, $5x \div 5 = 1x$, or x; and $x \div 2 = \frac{x}{2}$, written as a fraction.

and dividing these equal quantities by 9, the quotients must be equal, and we have, $x = \frac{72}{9} = 8$, the required term.

For another example, find the 2d term of the proportion $12 : x = 9 : 36$.

$$9 \times x = 12 \times 36$$

dividing by 9, $9 \times x = 432$

$$x = 48, \text{ the 2d term.}$$

In each of these examples, the unknown term might, by the changes allowed in proportions, have been put in the fourth place: $6 : 9 = x : 12$, is the same as $9 : 6 = 12 : x$, and $12 : x = 9 : 36$, might be written $9 : 36 = 12 : x$. It is usual, in stating a proportion, to put the unknown term always in the fourth place.

Application of the Principles of Proportion.

(78.) In all questions which admit of solution by proportion, we have given, the ratio between two quantities, and one term of an equal ratio between two other quantities.

Take for an example, the following problem :—

What is the price of 4 yards of cloth, when 12 yards are sold for 18 dollars?

Here the ratio is given between the length of the two pieces of cloth, and one term of the ratio of the corresponding prices; and it is evident that the ratio of the prices should be equal to the ratio of the lengths; or, in other words, that if one piece is twice, three times, or four times as long as the other, the price should be twice, three times, or four times as great. In stating a proportion from this problem, we have only to put these ratios equal, and arrange them so that the unknown term may be the fourth. We have, therefore, as 12 yards is to 4 yards, so is the price of 12 yards, namely, 18 dollars, to the price of 4 yards; or putting x for the unknown term, and writing it in the form of a proportion.

$$12 : 4 = 18 : x;$$

$$12x = 72$$

$$x = \frac{72}{12} = 6, \text{ the price of 4 yards.}$$

2. *If 21* men build 18 feet of wall in a given time, how *many feet* of wall will 31 men build in the same time?

Here, the ratio between the number of men must, evidently, be equal to the ratio between the quantities of work accomplished, and the problem gives the proportion,

$$\overset{\text{men.}}{21} : \overset{\text{men.}}{31} = \overset{\text{feet.}}{18} : \overset{\text{feet.}}{x}$$

$$21\,x = 18 \times 31 = 558$$

$$x = \frac{558}{21} = 26\tfrac{12}{21}, \text{ the answer.}$$

NOTE.—In stating a proportion, it is necessary to observe, that as the known term of the second ratio is always put for the antècedent, the corresponding term of the first ratio must also be the antecedent. In the first example, 18, the known term of the second ratio corresponds to 12, the antecedent of the first; 18 dollars being the price of 12 yards: and in the second example, 18 and 21 correspond to each other; 18 feet of wall being the work that 21 men will perform in the given time.

3. If I buy 368,21 acres of land for 12020,25 dollars, and sell 112,5 acres of the same land at the same price per acre, how much am I to receive for it?

$$\overset{\text{acres}}{368{,}21} : \overset{\text{acres.}}{112{,}5} = \overset{\text{dolls.}}{12020{,}25} : \overset{\text{dolls.}}{x}$$

$$368{,}21\,x = 112{,}5 \times 12020{,}25$$

$$368{,}21 \times x = 1352278{,}125$$

$$x = \frac{1352278{,}125}{368{,}21} = 3672{,}57. \text{ Ans.}$$

4. What is the wages of a man for 311 days, at 16 dollars per month of 30 days? Ans. $165,866.

5. How many yards of matting that is 3 feet broad will cover a floor that is 27 feet long and 20 feet broad?

Ans. 60.

6. The joint stock of two merchants is 13,150 dollars; they gain in trade 2600, what part of this belongs to A, whose capital is 8300?

$$13150 : 8300 = 2600 : x.$$

This proportion can be simplified by dividing the two first terms by 50(68), which gives, $263 : 166 = 2600 : x$; $x = 1641{,}06$, the answer.

Inverse Proportion.

(79.) Quantities are said to be in inverse proportion, when they are of such a nature that the ratio of any two of them diminishes, when the ratio of the other two increases.

For example; the greater the number of men engaged in a piece of work, the sooner it will be accomplished, and the faster a man travels, the less time will it take him to go a given number of miles.

For an example of this kind of proportion,

1. Let it be required to determine in what time 16 men will do a piece of work which 11 men can do in 24 days.

Evidently, the more the men employed, the shorter the time, and hence instead of stating the proportion,

$$11 : 16 = 24 : x;$$

the first ratio must be inverted, for since, from the nature of the problem, x must be less than 24, the consequent of the first ratio must be less than the antecedent, and the proportion must be written $16 : 11 = 24 : x$. Reducing this proportion we find for the fourth term $x = 16\frac{1}{2}$.

Take for another example;

2. Two men starting at the same time, walk a certain distance, A walks at the rate of 3 miles an hour, and arrives in 13 hours; in what time will B arrive who walks 4 miles an hour?

$$4 : 3 = 13 : x.$$

$x = 9\frac{3}{4}$ hours.

3. An engineer having raised 100 yards of a certain work in 24 days, with 5 men; how many must he employ to finish a like quantity of work in 15 days? Ans. 8.

4. A garrison of 536 men have provision for 12 months; how long will these provisions last if the garrison be increased to 1124 men? Ans. $174\frac{16}{281}$ days.

PROMISCUOUS EXAMPLES.

5. What is the interest of $275\frac{3}{4}$ dollars for a year at $5\frac{1}{4}$ per cent?

$$100 : 275\frac{3}{4} = 5\frac{1}{4} : x.$$

Reducing the fractions to decimals,

$$100 : 275{,}75 = 5{,}25 : x.$$

Dividing the first ratio by 25,

$$4 : 11{,}03 = 5{,}25 : x. \qquad x = 14{,}476875.$$

6. What length must be cut off a board $7\frac{3}{4}$ inches wide to contain a square foot? Ans. $18\frac{18}{31}$.

7. A wall that is to be built to the height of 36 feet, was raised 9 feet by 16 men in 6 days; how many men must be employed to finish it in 4 days at the same rate of working? Ans. 72.

8. The circumference of the earth is about 24877 miles; at what rate per hour is a person at the equator carried round, one whole revolution being made in 23 hours 56 minutes? Ans. $1039\frac{154}{359}$ miles.

COMPOUND PROPORTION.

(80.) A required quantity may be so related to other quantities, that in order to determine it, we are under the necessity of making use of two or more proportions. The following is a question of this kind:—

If the wages of 4 men, for 7 days, be 42 dollars, what will be the wages of 14 men for 9 days?

This question involves two proportions. Allowing the same number of men, the wages must be proportional to the number of days. This gives the proportion;

$$\overset{\text{days.}}{7} : \overset{\text{days.}}{10} = 42 : x$$
$$7 \times x = 420$$
$$x = \frac{420}{7} = 60, \text{ the wages of 4 men}$$

for 10 days. In the next place, the wages must be proportional to the number of men employed; and from this condition we have the proportion,

$$\overset{\text{men.}}{4} : \overset{\text{men.}}{14} = \overset{\text{dolls.}}{60} : x$$
$$4 \times x = 840$$
$$x = \frac{840}{4} = 210, \text{ the answer.}$$

(81.) This method may be abbreviated by the following

Rule.

Place that quantity which is of the same kind as the required quantity, in the 3d term of a proportion, and x *to denote the unknown term, in the 4th. Arrange above one another the several given ratios, according as they may be direct or inverse when compared with the ratio of the 3d term to the required term. Multiply these ratios together, term by term, continually. The result will be a simple proportion, which reduced, gives the answer.*

For an example, take the preceding problem; $42 : x$ will be the second ratio. The given ratios are, 1st, the ratio of the time, which is $\overset{\text{days.}}{7} : \overset{\text{days.}}{10}$; 2d, the ratio of the number of men, which is $4 : 14$. Arranging these according to the rule, we have

$$\left.\begin{matrix}7 : 10\\ 4 : 14\end{matrix}\right\} = 42 : x.$$

Multiplying term by term, the first ratios give $28 : 140 = 42 : x$.

Dividing the first ratio of this last proportion by 28, we have (68),

$$1 : 5 = 42 : x,$$

$$\text{whence, } x = 210, \text{ the answer.}$$

Take for another example, the following, in which several ratios are given, some direct, and some inverse.

If 180 men, in 6 days, when the days are 10 hours long, can dig a trench 200 yards long, 3 wide, and 2 deep; in how many days of 8 hours long, will 100 men dig a trench 360 yards long, 4 wide, and 3 deep?

$$\text{Ratio of the}\left\{\begin{array}{lr}\text{No. of men inverse,} & 100 : 180\\ \text{length of the trench direct,} & 200 : 360\\ \text{width of the trench direct,} & 3 : 4\\ \text{depth of the trench direct,} & 2 : 3\\ \text{length of the days inverse,} & 8 : 10\end{array}\right\} = 6 : x.$$

Multiplying according to the rule, $96000 : 777600 = 6 : x$.

By dividing both terms of the first ratio of this last proportion by 1200, we have,

$$80 : 648 = 6 : x.$$

Dividing antecedents by 2,

$$40 : 648 = 3 : x$$

$$\text{or, } 40 \times x = 1944$$

$$x = \frac{1944}{40} = 48{,}6 \text{ days, the ans.}$$

NOTE.—The first ratios may be reduced before multiplication, by dividing their terms by the same number; or since the two vertical columns, by the successive multiplication of their terms form a single ratio, we may divide by any number that will divide one term in each column, whether these terms belong to the same ratio or not. Reduced according to these principles, the foregoing example becomes,

$$\left.\begin{array}{r}5:9\\1:9\\1:1\\1:1\\2:1\end{array}\right\}=6:x$$

$$10:81=6:x$$

$$\text{or,}\quad 5:81=3:x$$

$$\text{whence,}\quad 5\times x=243$$

$$\text{and,}\quad x=\frac{243}{5}=48\tfrac{3}{5},\text{ or }48{,}6.$$

(82.) It will be observed in the first of the foregoing examples, that the answer or fourth term of the first proportion, is made the third of the second; and if the problem was of such a nature as to require another proportion, the fourth term of the second would be the third term of the next, and so on. To demonstrate the rule, therefore, it is only necessary to show, that in any series of proportions having this property, viz. that the 4th term of one becomes the third term of the next; the product of the first terms, will be to the product of the second terms, as the third term of the first proportion, is to the fourth term of the last.

To prove this, take first the numerical proportions,

$$2:4=3:6$$

$$\text{and}\ 5:10=6:12.$$

Multiplying these proportions term by term, we have,

$$5\times2:4\times10=3\times6:12\times6;$$

and dividing the second ratio of this last proportion by 6, we obtain

$$5\times2:4\times10=3:12.$$

Again, take the proportion $9 : 3 = 12 : 4$, and multiply the last proportion by it, term by term, we shall have

$$5\times2\times9 : 4\times10\times3 = 3\times12 : 4\times12$$

dividing the last ratio by 12, we have,

$$5\times2\times9 : 4\times10\times3 = 3 : 4.$$

Multiplying by the proportion $16 : 20 = 4 : 5$, and dividing the last ratio by 4, we have,

$$5\times2\times9\times16 : 4\times10\times3\times20 = 3 : 5.$$

Thus we might proceed with any number of proportions, for if the principle be true of two proportions, it must also be true of three, since the first two can be reduced to one, and compared with the third. If true of three, it must, for a like reason, be true of four, and so on.

To make this demonstration general, suppose we have,

$$a : b = c : d,$$
$$e : f = d : g,$$
$$h : i = g : l,$$
$$m : n = l : o,$$ we shall always have,

$$a\times e\times h\times m : b\times f\times i\times n = c : o.$$

For multiplying the first two proportions, term by term, we have,

$$a\times e : b\times f = c\times d : d\times g$$

and dividing the second ratio by d,

$$a\times e : b\times f = c : g.$$

Multiplying this proportion by the third, and dividing the last ratio by g, we have,

$$a\times e\times h : b\times f\times i = c\times l.$$

Multiplying by the fourth, and dividing the last ratio by l, we obtain the result,

$$a\times e\times h\times m : b\times f\times i\times n = c : o.$$

By proceeding in the same manner with any number of proportions, we should evidently obtain a like result. This is the same as taking the products of all the proportions which will give the same result. Taking the products of all the proportions, term by term, we have,

$$a\times e\times h\times m : b\times f\times i\times n = c\times d\times g\times l : d\times g\times l\times o;$$

and dividing the last ratio by $d \times g \times l$, we have as before,

$$a \times e \times h \times m : b \times f \times i \times n = c : o.$$

EXAMPLES FOR PRACTICE.

1. If a footman travel 130 miles in 3 days when the days are 12 hours long, in how many days of 10 hours each, may he travel 360 miles? Ans. $9\frac{63}{65}$ days.

2. If 100 dollars, in one year, gain 5 dollars interest, what will be the interest of 750 dollars for 7 years?
Ans. 262,50.

3. The interest of 2400 dollars for 5 years, at 6 per cent. is 720 dollars; at what rate of interest will 3000 dollars produce 816 dollars in 4 years. Ans. $6\frac{4}{5}$ per cent.

4. If 60 men can build 160 feet of wall, 6 feet thick, and 8 feet high in 6 days of 8 hours each, how many feet of wall 4 feet thick, and 6 feet high, will 25 men build in 7 days, each 9 hours long? Ans. 185 feet.

General Application of the Principles of Proportion.

(83.) 1. A sum of money is divided between two persons, so that the ratio of their shares is as 7 to 8, but the share of the one exceeds that of the other by $21; what was the amount of the whole sum, and each of the shares?

Let x represent the share of the first; then, since the other share exceeds it by $21, it will be $x+21$; then by the conditions of the question, we shall have the proportion,

$$8 : 7 = x + 21 : x.$$

Then, by subtracting consequents from antecedents, and comparing with consequents, we have (69),

$$8 - 7 : 7 = x + 21 - x : x,$$

or, $1 : 7 = 21 : x$,

whence, $x = 147$, the share of the first.

$147 + 21 = 168$, the other share, and $147 + 168 = 315$, the sum.

2. Two persons, A and B, travel the same road, A has 29 miles the start, and travels $3\frac{3}{4}$ miles an hour; B travels $4\frac{1}{3}$

miles an hour; how many miles must B travel before he overtakes A?

Let x be the distance that B travels; then the distance that A travels before B overtakes him will be $x-29$. But the distances must be proportional to the rate of travelling, hence,

$$4\tfrac{1}{3} : 3\tfrac{3}{4} = x : x-29,$$

$$\text{or,}\quad \frac{13}{3} : \frac{15}{4} = x : x-29.$$

Reducing the fractions of the first ratio to a common denominator, the proportion becomes,

$$\frac{52}{12} : \frac{45}{12} = x : x-29.$$

Multiplying the first ratio by 12 (67), we have,

$$52 : 45 = x : x-29.$$

Comparing the difference of antecedents and consequents with the antecedents, the proportion becomes (69),

$$52-45 : 52 = x-x+29 : x,$$

$$\text{or,}\quad 7 : 52 = 29 : x,$$

$$\text{whence,}\quad 7\times x = 1508,$$

$$\text{and,}\quad x = \frac{1508}{7} = 215\tfrac{3}{7}, \text{ the answer.}$$

3. Two numbers have such a relation to each other, that if 4 be added to each, they will be to each other as 3 to 4; and if 4 be subtracted from each, they will be to each other as 1 to 4. What are the numbers?

Let x be one of the numbers, and y the other, then we shall have the proportions,

$$x+4 : y+4 = 3 : 4,$$

$$\text{and,}\quad x-4 : y-4 = 1 : 4.$$

Taking the first proportion and comparing the antecedents with the difference of the antecedents and consequents, we have (69),

$$x+4 : y+4-x-4 = 3 : 1$$

$$\text{or,}\quad x+4 : y-x = 3 : 1.$$

From the second, by a like process, we obtain,

$$x-4 : y-4-x+4=1:3,$$

or, $$x-4 : y-x=1:3.$$

And dividing these results, term by term, we have (70),

$$\frac{x+4}{x-4}:1=\frac{3}{1}:\frac{1}{3}.$$

Multiplying the first ratio by $x-4$, and the second by 3 (67), we have,

$$x+4 : x-4=9:1.$$

And comparing the sum and difference of antecedents and consequents (69),

$$x+4-x+4 : x+4+x-4=8:10.$$

Reducing $$8:2x=4:5.$$

Dividing the antecedents by 4,

$$2:2x=1:5.$$

Dividing the first ratio by 2,

$$1:x=1:5;$$

wherefore, $$x=5.$$

To find the value of y, put this value of x in the first proportion; it will then become,

$$5+4:y+4=3:4,$$

or, $$9:y+4=3:4.$$

Changing extremes, $4:y+4=3:9$,

and, $$4:y+4-4=3:6,$$

or, $$4:y=3:6$$

$$4:y=1:2,$$

whence, $$y=8.$$

The numbers are therefore 5 and 8.

The sum of the two numbers, and their difference, are to each other in the ratio of 4 to 1, and their difference is to their product in the ratio of 1 to 15. Required the numbers,

$$\overset{\text{sum.}}{x+y} : \overset{\text{diff.}}{x-y}=4:1$$

$$\overset{\text{diff.}}{x-y} : \overset{\text{prod.}}{x\times y}=1:15.$$

From the first proportion we obtain (69)

$$x+y-x+y : x+y+x-y = 3 : 5$$

or, $$2y : 2x = 3 : 5$$

$$y : x = 3 : 5$$

$$y : x-y = 3 : 2 \text{ (69)},$$

or, $$x-y : y = 2 : 3 \text{ (66)}.$$

Then dividing the second proportion by this last,

$$\frac{x-y}{x-y} : \frac{x\times y}{y} = \frac{1}{2} : \frac{15}{3}$$

$$1 : x = \tfrac{1}{2} : 5$$

$$1 : x = 1 : 10$$

$x = 10$, the greater number.

To find y, we take the proportion,

$$y : x = 3 : 5,$$

or, $$y : 10 = 3 : 5$$

$$y : 2 = 3 : 1$$

$y = 6$, the less number.

ALGEBRA.

CHAPTER I.

DEFINITIONS, NOTATION, AND ALGEBRAIC OPERATIONS.

DEFINITIONS AND NOTATION.

(1.) ALGEBRA is that branch of mathematics, in which the letters of the alphabet, and other symbols are used, for the purpose of abridging and generalizing the reasoning required in the resolution of questions relating to numbers.

In the solution of algebraic problems, there are always certain quantities whose values are known, from which known quantities, it is proposed to determine others which are unknown; and it is usual to express known quantities by the first letters of the alphabet, as a, b, c, d, &c., and unknown quantities by the last, as x, y, z, t, u, &c.

The signs which denote algebraic operations and conditions are the same as in arithmetic, with the addition of the following:—

The sign of *inequality*, $>$, which is used to denote that one quantity is greater than another. The opening of the sign is always put towards the greater quantity, thus $a > b$ signifies that a is greater than b, and $a < b$ shows that a is less than b.

When quantities multiplied together, are represented by letters, the sign of multiplication is omitted. Thus, $a \times b \times c$ is written a, b, c: but when numbers are used, the sign is retained, for 5×6 is a very different thing from 56, on account of the local value of the figures.

For division, the sign $\frac{a}{b}$, or writing the numbers in a fractional form with the divisor beneath, is most commonly used.

The *coefficient* is a number written before a letter, to denote how many times this letter is taken in the expression, or, how many times + ONE, the quantity it expresses is added to itself. Instead of writing $a+a+a+a$, we write $4a$, to denote that a is taken 4 times; or which is the same thing added to itself 3 times: $ab+ab+ab$, is the same as $3ab$; $abc+abc$ is the same as $2abc$, and $\sqrt{a}+\sqrt{a}$ is the same as $2\sqrt{a}$. When no coefficient is written, 1 is always understood.

The *exponent* is a number written to the right of a letter, and a little above, to denote the number of times it is taken as a factor; or how many times + ONE it is multiplied by itself. Instead of writing $a \times a \times a \times a \times a$, or $aaaaa$, we write a^5, which is read *a, fifth power*, and signifies that a is a factor 5 times in the quantity; or that it is multiplied by itself 4 times. When no exponent is written, 1 is always understood.

(2.) Every quantity written with algebraic signs, is called an *algebraic quantity*. Thus, $3a$, $3a^4-b$, $4b^2c^2-a^3$, and $5b-2c+a$ are algebraic quantities.

When an algebraic quantity is not connected with any other by the sign of addition or subtraction, it is called a *monomial:* $3a$, $2b^2$, and $3c$ are monomials.

When it consists of two terms, it is a *binomial*, as $3a-b$, $4a^3+b^2$.

When it consists of three or more terms, it is called a *polynomial.*

The *numerical value* of an algebraic expression, is the number which would be obtained by giving to each letter that enters into it, its particular numerical value. Thus, the numerical value of the expression $3a+2b$, if we suppose $a=4$, and $b=7$, is $3\times4+2\times7$, or $12+14=26$. On the same supposition, $4a^2-b^2+2ab$ is equal to

$$4\times4^2-7^2+2\times4\times7=64-49+56=71.$$

The numerical value of a polynomial is not altered by changing the places of its terms. Thus,

$$4a^2+b-3a^2c-b^2+17abc, \text{ is the same as}$$
$$17abc-3a^2c+b+4a^2-b^2.$$

In any polynomial, the terms which have the sign + before them are called *additive* terms, and those which have the sign —, *subtractive* terms; when no sign is written, + is always understood. Additive terms are sometimes called *positive*, and subtractive terms, *negative* quantities.

The number of literal factors which enter into any term, is the *degree* of that term; ab is of the second *degree*, a^2b of the third, ab^2c^2 of the fifth. In general, the degree of an algebraic term is found by taking the sum of the exponents of the letters which enter into that term. An algebraic quantity which has all its terms of the same degree, is said to be *homogeneous*: $4a^5+2a^2b^3-3abc^3+b^4c$, is therefore homogeneous, and of the 5th degree.

Similar terms are those which contain the same letters in the same powers, $2a^2b$ and $5a^2b$ are similar. But, $3ab^2$ and $3ab$ are not similar terms, for the letters, although the same, are not in the same power.

If an algebraic quantity contain similar terms, it may be simplified or reduced. Thus, the quantity $7a^3b^2-4a^3b^2$ is evidently equal to $3a^3b^2$; $2ab+6ab$ is equal to $8ab$; and $16a^2b-8b^3+8b^3+8a^2b-21a^2b+3c^2$ is equal to $3a^2b+3c^2$; since $16a^2b+8a^2b=24a^2b$, and $24a^2b-21a^2b=3a^2b$.

A parenthesis ( ), indicates that whatever arithmetical operations are performed upon one of the letters contained within it, are to be performed upon the whole: thus, $(a-b)\times c$ shows that both a and b are to be multiplied by c; $a-(b+c)$ shows that b and c are both to be taken from a; $(a+b+c)(d+e)$ signifies that $a+b+c$ is to be multiplied by $d+e$; and $(a+b)^2$, $(abc)^3$, and $\left(\frac{c}{b}-\frac{c^2}{a}\right)^4$ indicates that $a+b$, abc, and $\frac{c}{b}-\frac{c^2}{a}$, are to be raised to their 2d, 3d and 4th powers, respectively. A vinculum or bar drawn over any number of letters, indicates the same as a parenthesis; thus, $\sqrt{\overline{a+x}}$ indicates the square root of the sum of a and x, $\sqrt{\overline{a-x}}$, the square root of the difference of a and x.

(3.) The following practical rules and exercises in algebraic notation will be of great use to the pupil before proceeding farther.

To write a quantity algebraically, express by means of algebraic symbols all the arithmetical operations and conditions belonging to that quantity.

EXAMPLES.

1. From the difference of the products of three times a multiplied by b, and 4 times a multiplied by the third power of c, subtract the quotient of b divided by c.

$$3ab - 4ac^3 - \frac{b}{c} \text{ is the algebraic expression.}$$

2. Write the following in algebraic language:—4 times the square root of the difference of a and c, added to the sum of twice a of the second power and b of the third power, is equal to $\frac{1}{2}$ the cube root of the fifth power of a, added to the third power of c.

Ans. $4\sqrt{a-c} + 2a^2 + b^3 = \frac{1}{2}\sqrt[3]{a^5} + c^3$.

3. Express in algebraic language, the quotient of 8 times the product of a and x subtracted from 9 times a of the third power multiplied by b of the second power, divided by the difference between a^2 and a.

Ans. $\dfrac{9a^3b^2 - 8ax}{a^2 - a}$.

4. Write the following:—

6 a diminished by x, the difference increased by the square root of $2a^4$, diminished by the cube root of 16, is equal to twice the product of $3a^3$ by b^2, diminished by the quotient of 21, divided by the square root of 6.

Ans. $6a - x + \sqrt{2a^4} - \sqrt[3]{16} = 6a^3b^2 - \dfrac{21}{\sqrt{6}}$.

5. Write the difference between the square root of the sum of the squares of a and b, and the square root of the difference of the squares of a and b.

Ans. $\sqrt{a^2+b^2} - \sqrt{a^2-b^2}$.

To find the value of an algebraic quantity.

(4.) *Substitute the numbers which the letters represent, and perform upon them all the arithmetical operations indicated.*

EXAMPLES.

1. Let $a=6$, $b=5$, $c=4$, and $d=1$, then will

$a^3 - 2ab + c - 3d = 6^3 - 2\times6\times5 + 4 - 3\times1$

$= 216 - 60 + 4 - 3 = 157$, the numerical value.

2. The numerical value of

$$a^2+3ab-c^2=36+90-16=110.$$

3. " " $\dfrac{a^3}{a+3c}+c^2=\dfrac{216}{18}+16=28.$

4. " " $\sqrt{2ac+c^2}=\sqrt{64}=8.$

5. " " $\sqrt{b^2-ac+\sqrt{2ac+c^2}}=3.$

6. " " $9ab-10b^2+c=16.$

7. " " $\dfrac{a^2-\sqrt{b^2-ac}}{2a-\sqrt{b^2+ac}}=7.$

Let $a=8$, $b=4$, $c=2$ and $d=\frac{1}{2}$.
Find the numerical value of

8. " " $\sqrt[3]{a^2}+3ab^3+\dfrac{c^6}{ad}.$

Ans. 1556.

9. " " $\dfrac{(a^2-b^2)\times(c^2-d^2)}{5abcd}-\dfrac{c^2+(4d)^2}{abc}.$

Ans. 1.

10. " " $(2b^3-\sqrt[3]{a^6})\times\dfrac{2\sqrt{b}}{2c}-2d^4.$

Ans. $63\frac{7}{8}$.

11. " " $\dfrac{a}{b^2}-3\sqrt{4cd}+11bc^3-\dfrac{a^2}{2d^3}.$

Ans. $90\frac{1}{2}$.

To reduce the similar terms of an algebraic expression to a single term.

(5.) *Add together the coefficients of all the additive quantities, and also the coefficients of the subtractive quantities, and take the difference between the sums : affect this difference with the sign which belonged to the greatest sum : this difference, written before the letters common to all the similar terms, will be the single term required.*

EXAMPLES.

1. Reduce $12a^3b^2-3ab^3-2a^3b^2-6a^3b^2-4ab^3+2ab^3+a^3b^2$. In this example we have 4 terms containing

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the letters a^3b^2, the same letters and the same powers of these letters. Two of these terms are additive, and two subtractive. The sum of the coefficients of the additive terms $12a^3b^2$ and a^3b^2, is 13. The sum of the coefficients of the subtractive terms $2a^3b^2$ and $6a^3b^2$, is 8; the difference of the sums is 5; and since the sum of the additive terms is greatest, the single term will have the sign +, and is $5a^3b^2$. There are also in this example, other terms, containing the literal quantity ab^3. These, reduced by the same rule, give $-5ab^3$, because the coefficients of the subtractive terms exceed those of the additive terms. The expression reduced becomes $5a^3b^2 - 5ab^3$.

2. Reduce $4a^2b - 8a^2b - 9a^2b + 11a^2b$.

Ans. $-2a^2b$.

3. Reduce $7abc^2 - abc^2 - 7abc^2 - 8abc^2 + 6abc^2$.

Ans. $-3abc^2$.

4. Reduce $4a^2c - 6b^3 + 18a^2c - 9b^3 - 13a^2c + 11b^3 - 16a^3b^2 + b^4$.

Ans. $9a^2c - 4b^3 - 16a^3b^2 + b^4$.

5. Reduce $a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3$.

Ans. $a^3 - b^3$.

6. Reduce $10a^4 - 15a^3x + 20a^2x^2 - 12a^3x + 18a^2x^2 - 24ax^3 - 4a^2x^2 + 6ax^3 - 8x^4$.

Ans. $10a^4 - 27a^3x + 34a^2x^2 - 18ax^3 - 8x^4$.

ADDITION OF ALGEBRAIC QUANTITIES.

RULE.

(6.) *Write down the quantities with their proper signs and reduce similar terms.*

EXAMPLES.

1. Find the sum of $3a^2b - 4b^3 + 6bc^2 - 8a^2c$, and $5b^3 + 9a^2c + 4a^2b - 10bc^2$.

The sum is $3a^2b - 4b^3 + 6bc^2 - 8a^2c + 5b^3 + 9a^2c + 4a^2b - 10bc^2$.

Reducing we have, $7a^2b + b^3 + a^2c - 4bc^2$.

NOTE.—It is convenient, in finding the sum of several polynomials, to write the similar terms one above another, as in the following example:—

2. Add together $15a^2 - 8b^2c + 32a^2c^3 - 12bc$, $19b^2c - 4a^2 + 11a^2c^3 + 2bc$, $a^2 - 29a^2c^3 - 12b^2c + 5bc$, and $9a^2c^3 - 14bc + b^2c$.

$$\begin{array}{rrrr}
15a^2 & -\ 8b^2c & +32a^2c^3 & -12bc \\
-4a^2 & +19b^2c & +11a^2c^3 & +\ 2bc \\
a^2 & -12b^2c & -29a^2c^3 & +\ 5bc \\
 & +b^2c & +\ 9a^2c^3 & -14bc \\
\hline
12a^2 & * \quad * & +23a^2c^3 & -19bc,
\end{array} \quad \text{the ans.}$$

3. What is the sum of $6ab + 12bc - 8cd$, $3cd - 7ab - 9bc$ and $12cd - 2ab - 5bc$?

Ans. $7cd - 3ab - 2bc$.

4. What is the sum of $5\sqrt{ab} - 7\sqrt{bc} + 8d$, $3\sqrt{ab} + 8\sqrt{bc} - 12d$ and $7\sqrt{ab} + 3\sqrt{bc} + 9d$?

Ans. $15\sqrt{ab} + 4\sqrt{bc} + 5d$.

5. What is the sum of $2a^2 + 3ab + 8c^2 + d^2$, $5a^2 - 7ab + 5c^2 - d^2$ and $4ab - 2a^2 + 3c^2 + 30$?

Ans. $5a^2 + 16c^2 + 30$.

6. What is the sum of $2a^2 - 3ab + 2b^3 - 3a^2$, $3b^3 - 2a^2 + a^3 - 5c^3$, $4c^3 - 2b^3 + 5ab + 100$ and $20ab + 16a^2 - bc - 8$?

Ans. $13a^2 + 22ab + 3b^3 + a^3 - c^3 + 92 - bc$.

8. What is the sum of

$$\begin{array}{l}
3a^2 + 9 + \sqrt{x} - 4 \\
2a - 8 + 2a^2 - 3x \\
4x^2 - 2a^2 + 18 - 7 \\
a - 12 - 3x^2 - 2y \\
\hline
\end{array}$$

Ans. $3a^2 + 3a + x^2 - 3x - 2y - 4 + \sqrt{x}$.

What is the sum of $14ax-2x^2$, $5ax-3xy$, $8y^2-4ax$ and $3x^2+26$?

Ans. $15ax+x^2-3xy+8y^2+26$.

Addition in algebra does not exactly correspond to addition in arithmetic. The operation sometimes includes both addition and subtraction; though in fact, it is only taking the sum of two quantities. Suppose it be required to add together the sum of $3a^2b$ and $2b^2$, and $4a^2b$, diminished by $2bc$. The first quantity would be written $3a^2b+2b^2$; adding to this $4a^2b$, we have, $3a^2b+2b^2+4a^2b$; but this quantity is too large, since $4a^2b$ was to be diminished by $2bc$; we have, therefore, to subtract $2bc$ from the sum, which gives $3a^2b+2b^2+4a^2b-2bc$, or by reducing, $7a^2b+2b^2-2bc$. This operation is in fact addition, though we both add and subtract; inasmuch as $4a^2b-2bc$ is an expression for the difference of two quantities, which difference was to be added to another quantity. Suppose $a=2$, $b=4$ and $c=1$: then $3a^2b+2b^2=80$, and $4a^2b-2bc=56$, and the sum of the numerical values is 136. But if c have such a value that $2bc>4ab$, the algebraic addition of two quantities, corresponds to the subtraction of the numerical values. For, if we suppose $c=9$, the other values remaining the same, $4a^2b-2bc=64-72=-8$; and hence, $80-8$ would be the numerical value of the algebraic sum, corresponding to an arithmetical difference.

This is the same as adding a subtractive quantity to an additive quantity. It is in effect subtraction. For example; adding $-b$ to a gives $a-b$, a quantity evidently less than a.

SUBTRACTION OF ALGEBRAIC QUANTITIES.

Rule.

(7.) *Change all the signs of the quantity to be subtracted, and then proceed as in addition, reducing similar terms to a single term.*

Suppose it be required to take $3a^2+3b$ from $16ab$: it is evident that the signs of $3a^2+3b$ are both to be changed; for if we take away $3a^2$, it will give for a remainder $16ab^2-3a^2$, but we have $3b$ also to take away from this remainder; since it was the sum of $3a^2$ and $3b$, that was to be taken from $16ab^2$; and this gives $16ab^2-3a^2-3b$.

Let it be required to subtract $3a^2 - 3b$ from $16ab^2$. Here it will be observed, that if we take $3a^2$ from $16ab^2$, the remainder will be too small by $3b$; for the expression $3a^2 - 3b$ shows that $3a^2$ is to be diminished by $3b$ before it is taken from $16ab^2$; $3b$ added to this remainder, will therefore give the true remainder.

This will evidently hold true whatever be the number of subtractive terms in the polynomial, since, in taking away the additive terms, we take away too much by the sum of all the subtractive terms, and hence, these must be added to the result.

EXAMPLES.

1. From $12a^3 - 2bc + 3c$, take $6a^3 + 4bc - 2c^2$,
Remainder, $6a^3 - 6bc + 3c + 2c^2$.

2. From $13a^2b^3 + 11a - 5a^2 + 6b$,
Take $7a - 5a^2 + 6b - 10a^2b^3$.
Remainder, $23a^2b^3 + 4a$.

3. From $3a + b + c - d - 10$,
Take $c + 2a - d$.
Remainder, $a + b - 10$.

4. From $3a + b + c - d - 10$,
Take $b - 19 + 3a$.
Remainder, $c - d + 9$.

5. From $2ab + b^2 - 4c + bc - b$,
Take $3a^2 - c + b^2$.
Remainder, $2ab - 3c + bc - 3a^2 - b$.

6. From $a^3 + 3b^2c + ab^2 - abc$,
Take $b^2 + ab^2 - abc$.
Remainder, $a^3 + 3b^2c - b^2$.

7. From $12x + 6a - 4b + 40$,
Take $4b - 3a + 4x + 6d - 10$.
Remainder, $8x + 9a - 8b - 6d + 50$.

8. From $2x - 3a + 4b - 16$,
Take $7x - 4b + 3a - 16$.
Remainder, $-5x - 6a + 8b$.

9. From $6a^2b + 7b^3 - a^2b^2$,
Take $13a^2b + 17b^3 - 21a^2b^2$.
Remainder, $20a^2b^2 - 7a^2b - 10b^3$.

From $27ax + 21b^2x - 3ab + 4b^2x^2 + 16ab^2 + 211$,
Take $19ax - 2b^2x + 6ab + 4b^2x^2 + 17ab^2 - 1$.

(8.) Note.—When one quantity is to be subtracted from another, the operation is sometimes indicated by writing this quantity included in a parenthesis, after the other, with the sign — before it, without changing the signs. Thus, $6a^3 - (7a^2 + 3ab - c)$ indicates that $7a^2 + 3ab - c$ is to be subtracted from $6a^3$. If the parenthesis be removed, the signs must be changed, since, in that case, the subtraction is actually performed. The quantity above would then become

$$6a^3 - 7a^2 - 3ab + c.$$

On this principle, polynomials may be made to undergo many transformations by including a number of terms in parentheses, changing their signs, and writing the sign — before them.

Thus, $6a^3 - 7a^2 - 3ab + c - 2ab^2$,
may be written, $6a^3 - (7a^2 + 3ab - c + 2ab^2)$,
or, $6a^3 - 7a^2 - (3ab - c + 2ab^2)$,
or, $6a^3 - 7a^2 - 3ab - (-c + 2ab^2)$.

By changing the order of the terms, we might have

$$c - 7a^2 + 6a^3 - 2ab^2 - 3ab,$$

which might again be subjected to a series of transformations, different from the former.

Let the pupil exercise himself in these transformations by writing the following polynomials in as many different forms as possible.

$16a^3 + bc^2 - 3a - 4bc + ab.$
$12a - 2b + 3ac^2 - 4a + c.$
$19a^2 - 36a + 7b - 17ab^2$
$a - b + 2d - c + ab.$

(9.) Subtraction in algebra, does not always imply diminution, for if we take $b - c$ from a, we shall have for a remainder, $a - b + c$. Now, if c be greater than b, the expression $a - b + c$ is evidently greater than a. But when c is greater than b, the quantity $b - c$, is a subtractive quantity; for, if we give b and c numerical values agreeing with this supposition, for example, $b = 6$, and $c = 8$, we shall have for the value of $b - c$, $6 - 8 = -2$. Let d be the difference between b and c, and let c be the greatest, then $b - c = -d$, and $-b + c = d$, and hence, $a - b + c = a + d$ (by putting d in the place of its equal $-b + c$); but $a - b + c$ results from the subtraction of $b - c = -d$ from a; hence, $-d$ being subtracted from a, the result is $a + d$, by which it will be seen, that the subtraction of a subtractive quantity, is the same as the addition of an additive one.

The sign —, before a monomial, shows that it is to be subtracted from some other quantity, and is easily understood when the monomial is preceded by a quantity greater than itself. But when this is not the case, or the subtractive quantity stands alone, as in the expressions -6 and -5. It may be well to inquire how it has been derived, and what is to be understood by it.

To illustrate this by numbers, let 5 be subtracted successively from 15; we shall have, $15 - 5 = 10$, $10 - 5 = 5$, $5 - 5 = 0$, and here the actual subtraction must stop, for the number 5, considered as the expression of a real quantity, cannot be taken from nothing. But, by means of the algebraic sign, we carry the subtraction still farther, and taking 5 from 0, we have for the result of the next subtraction, $0 - 5$, or simply, -5.

From the manner in which -5 has been derived, it is sometimes called a quantity less than nothing. But as no quantity can be *essentially* less than nothing, it is better to consider it only with reference to its algebraic sign, that is, such a quantity as is always to be subtracted whenever connected with other quantities, and which becomes 0, by the addition of an equal additive quantity.

Moreover, it is evident, that no error can arise from the introduction of such quantities into algebraic operations, inasmuch as their relation to other quantities may always be known: thus, -5 is such a quantity, that $+5$ being added to it, the sum will be 0; and as four successive subtractions of 5 from 15 give -5, so four successive additions of 5 to -5, will reproduce the number 15.

Subtractive quantities are often used to express conditions, or relations, opposite to those expressed by additive quantities. Thus, if we express a man's property in additive numbers, we can represent his debts by subtractive ones; for instance, if he have property worth \$500, and owes \$400, he is worth $500 - 400 = 100$ dollars; but if he owes \$600, he is worth

$500 - 600 = -100$ dollars, and may in some sense be said to be worth \$100 less than nothing; for if he were worth \$100 more, the value of his property would be nothing.

For another illustration, we might express a man's income by additive numbers, and his expenses by subtractive numbers. For, if we would know how much is added to his capital in a given time, we should subtract his expenditures from his income. If we suppose the latter \$1000, and the former \$900, there would be $1000 - 900 = 100$ dollars to be added to his capital; but if his expenditures were 1100, we should have $1000 - 1100 = -100$; or 100 dollars to be subtracted from his capital.

MULTIPLICATION OF ALGEBRAIC QUANTITIES.

MULTIPLICATION OF MONOMIALS.

Rule.

(10.) *Multiply the coefficients together, and write after their product all the letters which are common to both multiplier and multiplicand; affecting each letter with an exponent equal to the sum of the exponents of this letter in both factors. Letters not common to both factors, are to be written in the product with the same exponents with which they were affected in the factor to which they belonged.*

The reason for this rule will be evident from the principle laid down in arithmetic, that the product of one number by another may be obtained by multiplying it successively by all the factors of the multiplier (Arith. 10). For, let it be required to multiply $3a^3b^2$ by $8ab^3c$. By multiplying the coefficients together, we multiply by the factor 8, and we indicate the multiplication by each of the literal factors of the multiplier by writing them after, or with those of the multiplicand. As regards the exponents, let it be remarked, that $3a^3b^2$ is the same as $3aaabb$, and $8ab^3c$ is the same as $8abbbc$; and, as has been shown, the product of these two quantities will be $24aaaabbbbbc$; or, since $aaaa$ is equivalent to a^4, and $bbbbb$ to b^5, the product may be written $24a^4b^5c$, which is evidently the same result as would be obtained by adding the exponents according to the rule.

EXAMPLES.

1. Multiply $13a^4b^3c$ by $2ab$. Prod. $26a^5b^4c$.
2. Multiply $19a^2bcx$ by $7a^3bc^2$. Prod. $133a^5b^2c^3x$.
3. Multiply $72b^8c^2x$ by $\frac{1}{4}a^3bc^2x^3$. Prod. $18a^3b^9c^4x^4$
4. Multiply $9a^2x$ by $4x$. Prod. $36a^2x^2$.
5. Multiply $17a^3b^2c^3$ by $7ac$. Prod. $119a^4b^2c^4$.
6. Multiply $11a^5b^2c$ by $10a^5b^8c^9$. Prod. $110a^{10}b^{10}c^{10}$.
7. Multiply $121b^2c^3x$ by $5a^4bxy^2$. Prod. $605a^4b^3c^3x^2y^2$.
8. Multiply $77a^3cx^4$ by $61a^2b$. Prod. $4697a^5bcx^4$.
9. Multiply $117ab^2c^3x$ by $2a^3b^2c$. Prod. $234a^4b^4c^4x$.
10. Multiply $a^2b^3cx^2$ by b^2cx. Prod. $a^2b^5c^2x^3$.
11. Multiply $7a^2xy$ by $2ac^2x$. Prod. $14a^3c^2x^2y$.
12. Multiply $5abxy$ by a^2b^2y. Prod. $5a^3b^3xy^2$.
13. Multiply $15a^3c^2bxy$ by $9a^3cb^3y^2$. Prod. $135\,a^6c^3b^4xy^3$.

MULTIPLICATION OF POLYNOMIALS.

RULE.

(11.) *Multiply all the terms of the multiplicand by each term of the multiplier separately, observing that the product of any two terms which have like signs, that is, both* +, *or both* —, *must be affected with the sign* +, *and the product of any two terms having unlike signs, that is one* +, *and the other* —, *must be affected with the sign* —. *The sum of all these partial products will be the product of the given polynomials.*

1. Let it be required to multiply $a+b+f$ by $c+d$. The multiplication is indicated by writing the quantities in the following manner, $(a+b+f)\ (c+d)$.

Now, in multiplication, we propose to repeat the multiplicand as many times as there are units in the multiplier; and hence, in this example, we are to repeat $a+b+f$ as many times as there are units in $c+d$.

By taking each term c times, we repeat the multiplicand as many times as there are units in c: this gives $ac+bc+fc$. In like manner repeating it d times, we obtain, $ad+bd+fd$. Now, since the multiplicand was to be repeated $c+d$ times, it is evident, that the sum of these two partial products will be the whole product, and we have,

$$(a+b+f)(c+d)=ac+bc+fc+ad+bd+fd.$$

Hence it is evident, that to multiply polynomials, all the terms of which are additive, we are only to multiply all the terms of the multiplicand by each term of the multiplier, and take the sum of the products.

2. When some of the terms are additive and some subtractive, regard must be had to the signs; and we are next to prove, that like signs give a product affected with the sign $+$, and unlike signs, a product affected with the sign $-$.

To take the most general case: let all the additive terms of a polynomial be represented by a, and all the subtractive terms by $-b$; and let all the additive terms of another polynomial be represented by c, and the subtractive terms by $-d$; it is proposed to multiply these polynomials together. We shall have, $(a-b)(c-d)=ac-bc-ad+bd$, by the rule.

The correctness of the rule may be shown in the following manner:—when we multiply a by c, we take the multiplicand too great by b, we must, therefore, diminish the result by the product of b by c. This would give $ac-bc$ for the product of $a-b$ by c. But this product is evidently too great, for, in taking c for the multiplier, we have taken a quantity too great by d; hence it is necessary to multiply $a-b$ by d, and subtract the product. Now the product of $a-b$ by d, according to what has been before shown, is $ad-bd$. Subtracting this product from $ac-bc$, we have $ac-bc-ad+bd$, the signs of the latter product being changed according to the rule for subtraction.

It is, moreover, evident that the same reasoning which applies to $a-b$ and $c-d$, will apply to the polynomials, the additive and subtractive terms of which they represent.

To illustrate this, suppose we have the polynomials

$$2a^2b+3b^2-2ac-c^2,$$

$$\text{and} \quad 4b^2+a^2c-4c^2-ac.$$

We have taken $a=$ to the additive, and $b=$ to the subtractive terms of the multiplicand, $c=$ to the additive, and $d=$ to the subtractive terms of the multiplier. The product of the additive terms of the multiplicand by those of the multiplier will therefore correspond to ac, and will have the sign $+$; the product of the subtractive terms of the multiplicand by the additive terms of the multiplier, will correspond to *bc*, and will have the sign —; the product of the additive terms of the multiplicand by the subtractive terms of the multiplier, will correspond to *ad*, and will have the sign —; and the product of the subtractive terms of the multiplicand by those of the multiplier, will correspond to *bd*, and will have the sign +. This will give, throughout the product of terms of like signs, + and of terms of unlike signs, —. By performing the multiplication according to the rule, and separating he several products corresponding to *ac*, *bc*, *ad*, and *bd*, we shall have the following results :—

$$\begin{array}{ll}
\text{Multiplicand} & 2a^2b + 3b^2 - 2ac - c^2 \\
\text{Multiplier} & 4b^2 + a^2c - 4c^2 - ac
\end{array}$$

$$\begin{array}{r|l}
\hline
ac \left\{ \begin{array}{l} +8a^2b^3 + 12b^4 \\ +2a^4bc + 3a^2b^2c \end{array} \right. & \left. \begin{array}{l} -8ab^2c - 4b^2c^2 \\ -2a^3c^2 - a^2c^3 \end{array} \right\} bc \\
\hline
ad \left\{ \begin{array}{l} -8a^2bc^2 - 12b^2c^2 \\ -2a^3bc - 3ab^2c \end{array} \right. & \left. \begin{array}{l} +8ac^3 + 4c^4 \\ +2a^2c^2 + ac^3 \end{array} \right\} bd \\
\hline
\end{array}$$

It will be seen by inspecting the work, that the four products which correspond to *ac*, have each the sign +; those which correspond to *bc*, the sign —; those which correspond to *ad*, the sign —, and those which correspond to *bd*, the sign +.

In the multiplication of polynomials, it is usual to begin at the left, and write the first term of each partial product directly under the partial multiplier.

EXAMPLES.

$$\begin{array}{ll}
\text{1.—Multiplicand,} & 3a^2 - 2ab - b^2 \\
\text{Multiplier,} & 2a - 4b \\
\hline
 & 6a^3 - 4a^2b - 2ab^2 \\
 & \qquad - 12a^2b + 8ab^2 + 4b^3 \\
\hline
\text{Product,} & 6a^3 - 16a^2b + 6ab^2 + 4b^3
\end{array}$$

2.

$$\begin{array}{l} a^2+2ab+4b^2 \\ 2a^2-2ab+b^2 \\ \hline 2a^4+4a^3b+8a^2b^2 \\ \qquad -2a^3b-4a^2b^2-8ab^3 \\ \qquad\qquad\qquad +a^2b^2+2ab^3+4b^4 \\ \hline \end{array}$$

Prod. $2a^4+2a^3b+5a^2b^2+6ab^3+4b^4$

3.

$$\begin{array}{l} a^2+b^2+c^2-ab-ac-bc \\ a+b+c \\ \hline a^3+ab^2+ac^2-a^2b-a^2c-abc \\ \qquad a^2b+b^3+bc^2-ab^2-abc-b^2c \\ \qquad\qquad\qquad a^2c+b^2c+c^3-abc-ac^2-bc^2 \\ \hline \end{array}$$

Prod. $a^3+b^3+c^3-3abc$

4.

$$\begin{array}{l} a^3+2a^2b+2ab^2+b^3 \\ a^3-2a^2b+2ab^2-b^3 \\ \hline a^6+2a^5b+2a^4b^2+a^3b^3 \\ \quad -2a^5b-4a^4b^2-4a^3b^3-2a^2b^4 \\ \qquad\qquad +2a^4b^2+4a^3b^3+4a^2b^4+2ab^5 \\ \qquad\qquad\qquad\qquad -a^3b^3-2a^2b^4-2ab^5-b^6 \\ \hline \end{array}$$

Prod. a^6-b^6

5.

$$\begin{array}{l} 14ac-3ab+2 \\ ac-ab+1 \\ \hline 14a^2c^2-3a^2bc+2ac \\ \qquad -14a^2bc+3a^2b^2-2ab \\ \qquad\qquad\qquad +14ac-3ab+2 \\ \hline \end{array}$$

Prod. $14a^2c^2-17a^2bc+16ac+3a^2b^2-5ab+2.$

6. $$\begin{array}{l} x^2 - \frac{1}{2}x + \frac{2}{3} \\ \frac{1}{3}x + 2 \\ \hline \frac{1}{3}x^3 - \frac{1}{6}x^2 + \frac{2}{9}x \\ \qquad + 2x^2 - x + \frac{4}{3} \\ \hline \frac{1}{3}x^3 + \frac{11}{6}x^2 - \frac{7}{9}x + \frac{4}{3} \end{array}$$

Product, $\frac{1}{3}x^3 + \frac{11}{6}x^2 - \frac{7}{9}x + \frac{4}{3}$

7. Multiply $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.
Prod. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

8. Multiply $4x^2y + 3xy - 1$ by $2x^2 - x$.
Prod. $8x^4y + 2x^3y - 2x^2 - 3x^2y + x$.

9. Multiply $x^3 - x^2 + x - 5$ by $2x^2 + x + 1$.
Prod. $2x^5 - x^4 + 2x^3 - 10x^2 - 4x - 5$.

10. Multiply $3a^4 + 7a^3x + 13a^2x^2 + ax^3 - 4x^4$ by $a^2 - ax + x^2$.
Prod. $3a^6 + 4a^5x + 9a^4x^2 - 5a^3x^3 + 8a^2x^4 + 5ax^5 - 4x^6$.

11. Multiply $8a^3x^3 - 12a^2x^2 + 24ax - 8$ by $4a^2x^2 + 8ax + 4$.
Prod. $32a^5x^5 + 16a^4x^4 + 32a^3x^3 + 112a^2x^2 + 32ax - 32$.

12. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$.
Prod. $x^4 - y^4$.

13. Multiply $x^2 - \frac{3}{4}x + 1$ by $x^2 - \frac{1}{2}x$.
Prod. $x^4 - \frac{5}{4}x^3 + \frac{11}{8}x^2 - \frac{1}{2}x$.

14. Multiply $4x^2 - 2x + \frac{1}{4}$ by $4x^2 + 2x + \frac{1}{4}$.
Prod. $16x^4 - 2x^2 + \frac{1}{16}$.

15. Multiply $a^3 - 4a + 16a - 64$ by $a^2 + 4a + 16$.
Prod. $a^5 + 16a^3 - 64a^2 - 1024$.

16. Find the first four terms in the product of

$$a^m + a^{m-1}x + a^{m-2}x^2 + a^{m-3}x^3 + \&c. \text{ and } a + x.$$

$$\begin{array}{l} a^m + a^{m-1}x + a^{m-2}x^2 + a^{m-3}x^3 + \&c. \\ a + x \\ \hline a^{m+1} + a^m x + a^{m-1}x^2 + a^{m-2}x^3 + \&c. \\ \qquad + a^m x + a^{m-1}x^2 + a^{m-2}x^3 + \&c. \\ \hline a^{m+1} + 2a^m x + 2a^{m-1}x^2 + 2a^{m-2}x^3 + \&c. \end{array}$$

17. Multiply $a^n + b^n$ by $a^2 + b^2$

$$\begin{array}{lll}
a^n + b^n & & \\
a^2 + b^2 & & \\
\hline
a^{n+2} + a^2b^n & & \\
 & a^nb^2 + b^{n+2} & \\
\hline
a^{n+2} + a^2b^n + a^nb^2 + b^{n+2} & &
\end{array}$$

18. Multiply $a^n + a^{n-1}x + a^{n-2}x^2 + a^{n-3}x^3$ &c. by $a - x$.

$$\begin{array}{llllll}
a^n & + a^{n-1}x & + a^{n-2}x^2 & + a^{n-3}x^3 & + \&c. & \\
a - x & & & & & \\
\hline
a^{n+1} & + a^nx & + a^{n-1}x^2 & + a^{n-2}x^3 & + \&c. & \\
 & - a^nx & - a^{n-1}x^2 & - a^{n-2}x^3 & + a^{n-3}x^4 & + \&c. \\
\hline
a^{n+1} & \text{"} & \text{"} & \text{"} & + a^{n-3}x^4 &
\end{array}$$

(12.) It is sometimes convenient to have the product of two algebraic quantities arranged according to the powers of some particular letter, so that the highest power of this letter may stand at the left, and the next highest, one term to the right, and so on. The following is a polynomial arranged with reference to the powers of a.

$$a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$$

We might also arrange it with reference to the powers of x, and then it would stand thus,

$$x^4 + 4ax^3 + 6a^2x^2 + 4a^2x + a^4.$$

When more than one term of the polynomial contains the same power of the leading letter, it is customary to write these terms, one above another, on the left of a vertical line, and the letter on the right, as in the following example:—

The polynomial $11a^2b - 19abc + 10a^3 - 15a^2c + 3ab^2 + 15bc^2 - 5b^2c$, arranged with reference to the powers of a, is written thus:

$$\begin{array}{r|l|l}
10a^3 + 11b & a^2 + 3b^2 & a - 5b^2c + 15bc^2. \\
-15c & -19bc &
\end{array}$$

19. Multiply $a + x + x^2 + x^3 + x^4$ by $a + x$, and arrange the product with reference to the powers of x.

$$\begin{array}{l}
a+x+x^2+x^3+x^4 \\
a+x \\
\hline
a^2+ax+ax^2+ax^3+ax^4 \\
\qquad ax+x^2\ \ +x^3\ \ +x^4+x^5 \\
\hline
a^2+2ax+ax^2+x^2+ax^3+x^3+ax^4+x^4+x^5
\end{array}$$

Product arranged,

$$\begin{array}{r|r|r|l}
x^5+a & x^4+a & x^3+a & x^2+2ax+a^2 \\
+1 & +1 & +1 &
\end{array}$$

20. Multiply $4a^2b-3a^2c+5ab^2-3ac^2$ by $3ab-5ac+2b^2$, and arrange the product with reference to the powers of a.

Product arranged,

$$\begin{array}{r|r|r|l}
12b^2 & a^3+23b^3 & a^2+10b^4 & a \\
-29bc & -31b^2c & -6b^2c^2 & \\
+15c^2 & -9bc^2 & & \\
 & +15c^3 & &
\end{array}$$

DIVISION OF ALGEBRAIC QUANTITIES.

(13.) Division is the opposite of multiplication, for its object is, when the product of two factors, and one of these factors, are given, to find the other factor. We have, therefore, only to reverse the rule for multiplication.

For the division of monomials, we have then the following

Rule.

Divide the coefficient of the dividend by the coefficient of the divisor, and to the quotient annex the letters common to both, each affected with an exponent equal to the difference of the exponents of this letter in the dividend and divisor. Letters found only in the dividend, are to be written with the same exponents in the quotient.

Examples.

1. Divide $48a^3b^4c^2$ by $12a^2b$.

Here 4 is the quotient of the coefficients, and taking the exponents of the divisor from the exponents of the same letters in the dividend, gives ab^2c^2; hence, $4ab^2c^2$ is the required quotient.

2. Divide $66a^3b^2c$ by $6ab^2c$.*	Quo. $11a^2$.
3. Divide $117a^2x^2$ by $39ax$.	Quo. $3ax$.
4. Divide $75a^4b^2c^3d$ by $5ab^2c$.	Quo. $15a^3c^2d$.
5. Divide $256a^6b^4cd^3$ by $16b^3cd^2$.	Quo. $16a^6bd$.

(14.) When the coefficients are not divisible, when the divisor contains letters not in the dividend, or when the exponent of any letter of the divisor is greater than the exponent of the same letter in the dividend, the division cannot be performed, and must be indicated by writing the quantities in a fractional form.

Thus, $27ab^3c^2$ divided by $15a^2bd$, must be written

$$\frac{27ab^3c^2}{15a^2bd}.$$

This fraction may be reduced by dividing both numerator and denominator by any quantity, that would divide both exactly (Arith. 25). In the example above, $3ab$ is the common divisor of both terms of the fraction, and dividing both terms by this gives

$$\frac{9b^2c^2}{5ad}.$$

This same operation may be performed more conveniently by the following

RULE.

Reduce the fractional coefficient to its lowest terms, and take the difference of the exponents of letters common to both terms of the fraction; affect each of the common letters with an exponent equal to this difference, and write it in that term of the fraction in which it had the highest exponent.

EXAMPLES.

1. Divide $48a^3bc^2d$ by $20ab^2c^3$.

$$\text{Ans. } \frac{48a^3bc^2d}{20ab^2c^3} = \frac{12a^2d}{5bc}.$$

---

* *Since* b^0 and c^0 are each equal to unity, they are omitted in the product, for $11a^2$ *is the same as* $11a^2 \times b^0 \times c^0$, or $11a^2 \times 1 \times 1$.

2. Divide $117a^5b^4c^3$ by $78a^5bc^4$.

Ans. $\dfrac{3b^3}{2c}$.

3. Divide $96abc$ by $12a^3bc^2d$.

Ans. $\dfrac{8}{a^2cd}$.

4. Divide a^5bc^2 by $a^6b^2c^4$.

Ans. $\dfrac{1}{abc^2}$.

5. Divide $27a^3b^4cd^2$ by $21abcd$.

Ans. $\frac{9}{7}a^2b^3d$.

6. Divide $14ab^2cd$ by $6a^2bc^2$.

Ans. $\dfrac{7bd}{3ac}$.

DIVISION OF POLYNOMIALS.

To divide a Polynomial by a Monomial.

RULE.

(15.) *Divide each term of the dividend by the divisor, according to the preceding rule, observing, that when the signs of the divisor and dividend are alike, the quotient must have the sign* $+$; *but if they be unlike, the quotient must have the sign* $-$.

Let it be required to divide $12a^2b^2 + 16a^3b^3c + 24ab^4c^2$ by $4ab^2$.

Since in multiplication each term of the multiplicand is multiplied separately by the multiplier, it is evident, that if we have the product and multiplier, which, in the case of division, become the dividend and divisor, we are only to reverse the process to find the multiplicand, or quotient; and hence, each term of the dividend is to be divided by the divisor; therefore, $12a^2b^2 + 16a^3b^3c + 24ab^4c^2 \div 4ab^2$, gives, for a quotient, $3a + 4a^2bc + 6b^2c^2$.

With regard to the signs, let it be remarked, that the signs of the several terms of the quotient must be such, that if it be multiplied by the divisor, the signs of the product will be the same as those of the dividend; for the quotient multiplied by the divisor should reproduce the dividend. We shall have, therefore :—

$$+ab \div +b = +a, \text{ for } +a \times +b = +ab;$$
$$-ab \div -b = +a, \text{ for } +a \times -b = -ab;$$
$$-ab \div +b = -a, \text{ for } -a \times +b = -ab;$$
$$+ab \div -b = -a, \text{ for } -a \times -b = +ab;$$

from which it will be seen, that in division, as well as in multiplication, like signs give +, and unlike signs —.

EXAMPLES.

1. Divide $130a^2b^3c + 15b^3c^2 - 25ab^2c^3 - 30bc^2$ by $5bc$.

$$5bc)\underline{130a^2b^3c + 15b^3c^2 - 25ab^2c^3 - 30bc^2}$$
$$26a^2b^2 + 3b^2c - 5abc^2 - 6c \quad \text{the quo.}$$

2. Divide $175a^3x^3 - 110a^2x^4 - 15ax^5$ by $5ax^2$.

Quo. $35a^2x - 22ax^2 - 3x^3$.

3. Divide $12a^4x^2 - 9a^3bx + 14a^2b^2 - 4ab^3$ by $4ab$.

Quo. $\frac{3a^3x^2}{b} - \frac{9a^2x}{4} + \frac{7}{2}ab - b^2$.

4. Divide $22a^2b^2c - 33a^4b^2c^2 + 41ab^4c^3 - 55ab^5c^4$ by $-11a^2b^2c$.

Quo. $-2 + 3a^2c - \frac{41b^2c^2}{11a} + \frac{5b^3c^3}{a}$.

5. Divide $15a^2bc - 15acx^2 + 5ad^2$ by $-5ac$.

Quo. $-3ab + 3x^2 - \frac{d^2}{c}$.

6. Divide $10a^2x - 15x^2 - 25x$ by $5x$.

Quo. $2a^2 - 3x - 5$.

7. Divide $-20ab + 60ab^3$ by $-6ab$.

Quo. $\frac{10}{3} + 10b^2$.

8. Divide $6abc + 12abx + 9a^2b$ by $3ab$.

9. Divide $16a^3b^2x - 64a^4b^3c^2 + 256a^5b^4c^3$ by $8a^2bc$.

10. Divide $212axy + 39a^2x^3y - 5a^4xy^3$ by $41a^2xy$.

To divide one Polynomial by another.

RULE.

(16.) 1*st. Arrange the dividend and divisor according to the powers of the same letter.*

2. *Divide the first term of the dividend by the first term of the divisor, and set the result in the quotient.*

3. *Multiply the whole divisor by the quotient thus found, and subtract the product from the dividend.*

The remainder will form a new dividend, with which proceed as before, till the first term of the divisor is no longer contained in the first term of the remainder.

The divisor and remainder, if there be a remainder, are then to be written in the form of a fraction, as in division of numbers.

EXAMPLES.

1. Divide $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$ by $a^2 - 2ab + b^2$.

$$\begin{array}{lll}
 & \text{Dividend.} & \text{Divisor.} \\
 & a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 & \left|\, a^2 - 2ab + b^2 \right. \\
 & a^4 - 2a^3b + a^2b^2 & \left|\, \overline{a^2 - 2ab + b^2} \right. \text{ Quo.} \\
\hline
\text{1st Rem.} & -2a^3b + 5a^2b^2 - 4ab^3 & \\
 & -2a^3b + 4a^2b^2 - 2ab^3 & \\
\hline
\text{2d Rem.} & \qquad a^2b^2 - 2ab^3 + b^4 & \\
 & \qquad a^2b^2 - 2ab^3 + b^4 & \\
\hline
\end{array}$$

The reason for the rule given above will appear from an examination of the example. The dividend being the product of two factors, one of which is the divisor, and the other the quotient sought, we may consider it as having been produced by their multiplication. Now, since the dividend is arranged with reference to the powers of a, it is evident, that the first term, which has the highest power of a, must have been produced by the multiplication of the terms having the highest powers of a in the divisor and quotient; and for this reason could be similar to no other term in the product, and could have been reduced with no other. It is, therefore, the product of the first term of the divisor by the first term of the quotient, when arranged with reference to the powers of a. The first term of the quotient will therefore be found by dividing the first term of the dividend by the first term of the divisor

2. As the dividend is the sum of all the partial products formed by multiplying the divisor by each term of the quotient, if we multiply the divisor by the term of the quotient thus found, and subtract the product from the dividend, the remainder will be the sum of the partial products formed by the divisor and the remaining terms of the quotient. We have therefore a new dividend, the first term of which, having the highest power of a, must have resulted from the multiplication of the first term of the divisor by the first term of the remaining part of the quotient. It is evident, therefore, that the second term of the quotient will be found by dividing the first term of the new dividend by the first term of the divisor. The same reasoning will apply to all the remaining terms of the dividend.

If the divisor, in the given example, be multiplied by the quotient, it will be seen that the first term in each of the partial products, corresponds to the first terms of the several dividends or remainders.

$$\begin{array}{lllll}
a^2 - 2ab + b^2 & & & & \\
a^2 - 2ab + b^2 & & & & \\
\hline
a^4 & - 2a^3b & + a^2b^2 & & \\
 & - 2a^3b & + 4a^2b^2 & - 2ab^3 & \\
 & & + a^2b^2 & - 2ab^3 & + b^4 \\
\hline
a^4 & - 4a^3b & + 6a^2b^2 & - 4ab^3 & + b^4
\end{array}$$

If, instead of arranging the given polynomials with reference to the powers of a, they had been arranged with reference to b, the result would have been the same, except in the order of the terms of the quotient.

$$\begin{array}{lllll}
b^4 & - 4ab^3 & + 6a^2b^2 & - 4a^3b & + a^4 \\
b^4 & - 2ab^5 & + a^2b^2 & & \\
\hline
 & - 2ab^3 & + 5a^2b^2 & - 4a^3b & \\
 & - 2ab^3 & + 4a^2b^2 & - 2a^3b & \\
\hline
 & & a^2b^2 & - 2a^3b & + a^4 \\
 & & a^2b^2 & - 2a^3b & + a^4 \\
\hline
\end{array}
\left\{
\begin{array}{l}
b^2 - 2ab + a^2 \\
\hline
b^2 - 2ab + a^2
\end{array}
\right.$$

2. Divide $a^6 - b^6$ by $a^3 - 2a^2b + 2ab^2 - b^3$.

$$
\begin{array}{ll}
a^6 \quad\text{———}\quad b^6 & \big)\; a^3 - 2a^2b + 2ab^2 - b^3 \\
a^6 - 2a^5b + 2a^4b^2 - a^3b^3 & \big(\; \overline{a^3 + 2a^2b + 2ab^2 + b^3} \text{ quotient.} \\
\hline
+2a^5b - 2a^4b^2 + a^3b^3 - b^6 & \\
+2a^5b - 4a^4b^2 + 4a^3b^3 - 2a^2b^4 & \\
\hline
\qquad 2a^4b^2 - 3a^3b^3 + 2a^2b^4 - b^6 & \\
\qquad 2a^4b^2 - 4a^3b^3 + 4a^2b^4 - 2ab^5 & \\
\hline
\qquad\qquad a^3b^3 - 2a^2b^4 + 2ab^5 - b^6 & \\
\qquad\qquad a^3b^3 - 2a^2b^4 + 2ab^5 - b^6 & \\
\hline
\qquad\qquad 0 \quad\; 0 \quad\; 0 \quad\; 0 &
\end{array}
$$

3. Divide $x^6 - x^4 + x^3 - x^2 + 2x - 1$ by $x^2 + x - 1$.

$$
\begin{array}{ll}
x^6 - x^4 + x^3 - x^2 + 2x - 1 & \big)\; x^2 + x - 1 \\
x^6 + x^5 - x^4 & \big(\; \overline{x^4 - x^3 + x^2 - x + 1} \\
\hline
\quad -x^5 + x^3 - x^2 & \\
\quad -x^5 - x^4 + x^3 & \\
\hline
\qquad +x^4 - x^2 + 2x & \\
\qquad\; x^4 + x^3 - x^2 & \\
\hline
\qquad\qquad -x^3 + 2x - 1 & \\
\qquad\qquad -x^3 - x^2 + x & \\
\hline
\qquad\qquad\quad +x^2 + x - 1 & \\
\qquad\qquad\quad\; x^2 + x - 1 &
\end{array}
$$

4. Find four terms of the quotient of $a^{m+1} + 2a^mx + 2a^{m-1}x^2 + 2a^{m-2}x^3 + \&c.$ divided by $a + x$.

$$
\begin{array}{ll}
a^{m+1} + 2a^mx + 2a^{m-1}x^2 + 2a^{m-2}x^3 + \&c. & \big)\; a + x \quad [+a^{m-3}x^3 \;\&c. \\
a^{m+1} + a^mx & \big(\; \overline{a^m + a^{m-1}x + a^{m-2}x^2} \\
\hline
\qquad a^mx + 2a^{m-1}x^2 & \\
\qquad a^mx + a^{m-1}x^2 & \\
\hline
\qquad\qquad a^{m-1}x^2 + 2a^{m-2}x^3 & \\
\qquad\qquad a^{m-1}x^2 + a^{m-2}x^3 & \\
\hline
\qquad\qquad\qquad a^{m-2}x^3 + \&c. &
\end{array}
$$

5. Divide $a^2 - b^2$ by $a - b$.

$$\begin{array}[t]{l} a^2 - b^2 \\ a^2 - ab \\ \hline ab - b^2 \\ ab - b^2 \\ \hline \end{array} \left.\right\} \begin{array}[t]{l} a - b \\ \hline a + b \text{ quotient.} \end{array}$$

6. Divide $a^3 - b^3$ by $a - b$.

$$\begin{array}[t]{l} a^3 - b^3 \\ a^3 - a^2b \\ \hline a^2b - b^3 \\ a^2b - ab^2 \\ \hline ab^2 - b^3 \\ ab^2 - b^3 \\ \hline \end{array} \left.\right\} \begin{array}[t]{l} a - b \\ \hline a^2 + ab + b^2 \end{array}$$

7. Divide $a^4 - b^4$ by $a - b$.

$$\begin{array}[t]{l} a^4 - b^4 \\ a^3 - a^3b \\ \hline a^3b - b^4 \\ a^3b - a^2b^2 \\ \hline a^2b^2 - b^4 \\ a^2b^2 - ab^3 \\ \hline ab^3 - b^4 \\ ab^3 - b^4 \\ \hline \end{array} \left.\right\} \begin{array}[t]{l} a - b \\ \hline a^3 + a^2b + ab^2 + b^3 \end{array}$$

8. Divide $a^5 - b^5$ by $a - b$.

$$\begin{array}[t]{l} a^5 - b^5 \\ a^5 - a^4b \\ \hline a^4b - b^5 \\ a^4b - a^3b^2 \\ \hline a^3b^2 - b^5 \\ a^3b^2 - a^2b^3 \\ \hline a^2b^3 - b^5 \\ a^2b^3 - ab^4 \\ \hline ab^4 - b^5 \\ ab^4 - b^5 \\ \hline \end{array} \left.\right\} \begin{array}[t]{l} a - b \\ \hline a^4 + a^3b + a^2b^2 + ab^3 + b^4 \end{array}$$

9. Divide $a^m - b^m$ by $a - b$.

$$\begin{array}{l}
\begin{array}{l} a^m - b^m \\ a^m - a^{m-1}b \end{array} \left\} \begin{array}{l} a - b \\ \hline a^{m-1} + a^{m-2}b + a^{m-3}b^2 + a^{m-4}b^3 + a^{m-5}b^4 + a^{m-6}b^5 \end{array} \right. \\ \hline
\quad a^{m-1}b - b^m \\
\quad a^{m-1}b - a^{m-2}b^2 \\ \hline
\qquad a^{m-2}b^2 - b^m \\
\qquad a^{m-2}b^2 - a^{m-3}b^3 \\ \hline
\qquad\quad a^{m-3}b^3 - b^m \\
\qquad\quad a^{m-3}b^3 - a^{m-4}b^4 \\ \hline
\qquad\qquad a^{m-4}b^4 - b^m \\
\qquad\qquad a^{m-4}b^4 - a^{m-5}b^5 \\ \hline
\qquad\qquad\quad a^{m-5}b^5 - b^m \\
\qquad\qquad\quad a^{m-5}b^5 - a^{m-6}b^6 \\ \hline
\qquad\qquad\qquad a^{m-6}b^6 - b^m \\ \hline
\end{array}$$

10. Divide

$$10a^3 \begin{array}{r|} +11b \\ -15c \end{array} a^2 \begin{array}{r|} +3b^2 \\ -19bc \end{array} a - 5b^2c + 15bc^2 \text{ by } 5a^2 + 3ba - 5bc.$$

$$\begin{array}{ll}
 & 10a^3 \begin{array}{r|} +11b \\ -15c \end{array} a^2 \begin{array}{r|} +3b^2 \\ -19bc \end{array} a - 5b^2c + 15bc^2 \left\{ \begin{array}{l} 5a^2 + 3ba - 5bc \\ \hline 2a + b - 3c \end{array} \right. \\ \cline{2-2}
 & 10a^3 + 6b \,|\, a^2 - 10bc \,|\, a \\ \cline{2-2}
\textbf{1st Rem.} & \begin{array}{r|} +5b \\ -15c \end{array} a^2 \begin{array}{r|} +3b^2 \\ -9bc \end{array} a - 5b^2c + 15bc^2 \\ \cline{2-2}
 & +5b \,|\, a^2 + 3b^2 \,|\, a - 5b^2c \\ \cline{2-2}
\textbf{2d Rem.} & -15c \,|\, a^2 - 9bc \,|\, a + 15bc^2 \\ \cline{2-2}
 & -15c \,|\, a^2 - 9bc \,|\, a + 15bc^2 \\ \cline{2-2}
 & \qquad\quad 0 \qquad\qquad 0
\end{array}$$

11. Divide $12a^3b^2 - 29a^3bc + 15a^3c^2 + 23a^2b^3 - 31a^2b^2c - 9a^2bc^2 + 15a^2c^3 + 10ab^4 + 6ab^2c^2$ by $3ab - 5ac + 2b^2$.

Dividend arranged

$$
\begin{array}{r|l r|l r|l}
12b^2 & a^3 + & 23b^3 & a^2 + & 10b^4 & a \\
-29bc & & -31b^2c & & -6b^2c^2 & \\
+15c^2 & & -9bc^2 & & & \\
 & & +15c^3 & & & \\
\hline
12b^2 & a^3 & 8b^3 & a^2 & & \\
-20bc & & -6b^2c & & & \\
-9bc & & & & & \\
+15c^2 & & & & & \\
\hline
\text{1st Rem.} & & 15b^3 & a^2 + & 10b^4 & a \\
 & & -25b^2c & & -6b^2c^2 & \\
 & & -9bc^2 & & & \\
 & & +15c^3 & & & \\
\hline
 & & 15b^3 & a^2 & 10b^4 & a \\
 & & -25b^2c & & -6b^2c^2 & \\
 & & -9bc^2 & & & \\
 & & +15c^3 & & & \\
\hline
\text{2d Rem.} & & 0 & & 0 &
\end{array}
$$

Divisor arranged.

$$
\begin{array}{r|l r|l}
3b & a + 2b^2 & & \\
-5c & & & \\
\hline
4b & a^2 + & 5b^2 & a \\
-3c & & -3c^2 &
\end{array}
\quad \text{quotient.}
$$

Product of the divisor by the first term of the quotient.

Product of the divisor by the 2d term of the quo.

In this example, when we wish to find the quotient of the first term of the dividend by the first term of the divisor, we first consider that a^3 divided by a, gives a^2; we then divide the quantities which stand at the left of the vertical line in the dividend, and are understood to be multiplied by a^3, by the quantities multiplied into a in the divisor; the quotient is that *part* of the general quotient which is multiplied by a^2. The partial quotient so obtained is then multiplied into the whole of the divisor, and the product subtracted from the whole dividend.

In the preceding example, we take the quantity $12b^3 - 29bc + 15c^2$, which is multiplied by a^3, and divide it by $3b - 5c$, which in the divisor is multiplied by a.

$$
\begin{array}{r|l}
12b^3 - 29bc + 15c^2 & 3b - 5c \\
\cline{2-2}
12b^2 - 20bc & 4b - 3c \\
\cline{1-1}
-9bc + 15c^2 & \\
-9bc + 15c^2 &
\end{array}
$$

Performing the operation, we have $4b - 3c$; and since a^3, *divided* by a, gives a^2, this quantity is to be multiplied by a^2; and hence the first term of the quotient is $\begin{array}{r|l} 4b & a^2. \\ -3c & \end{array}$

After the multiplication of the divisor by this term, and the subtraction of the product from the dividend, we have, for the first term of the new dividend, certain quantities multiplied into a^2; a^2 divided by a gives a; and dividing the quantities multiplied by a^2 by those multiplied by a, we have,

$$\begin{array}{l|l}
15b^3 - 25b^2c - 9bc^2 + 15c^3 & 3b - 5c \\ \cline{2-2}
15b^2 - 25b^2c & 5b^2 - 3c^2 \\ \cline{1-1}
\qquad\qquad -9bc^2 + 15c^3 & \\
\qquad\qquad -9bc^2 + 15c^3 & \\ \cline{1-1}
\end{array}$$

$\left.\begin{array}{r} 5b^2 \\ -3c^2 \end{array}\right| a$ is therefore the second term of the quotient. Multiplying this into the whole divisor, and subtracting the product from the second dividend, 0 remains; and hence the division is completed.

12. Divide $12a^4b - 26a^4b^2 + 10a^4b^3 + 18a^3b^3 - 19a^3b - 30a^3b^4 + 25a^3b^2 + 8a^3 + 9a^2b^3 - 15a^2b^4 + 12a^2b^2 - 10a^2 + 6ab^2$ by $3a^2b - 5a^2b^2 + 20$.

Dividend arranged. Divisor arranged.

$$\begin{array}{r|r|r|l}
12b & a^4 + 18b^3 & a^3 + \quad 9b^3 & \\
-26b^2 & -19b & -15b^4 & a^2 + 6ab^2 \\
+10b^3 & -30b^4 & +12b^2 & \\
 & +25b^2 & -10 & \\
 & +8 & & \\ \hline
12b & a^4 + 8 & a^3 & \\
-20b^2 & -4b & & \\
-6b^2 & & & \\
+10b^3 & & & \\ \hline
\end{array}
\qquad
\left\{\begin{array}{r|l|l}
3b & a^2 & \\
-5b^2 & \quad +2a & \\ \hline
4 & a^2 + 6b^2 & a \\
-2b & \quad -5 & +3b^2
\end{array}\right.$$

$$\text{1st Rem.}\left\{\begin{array}{r|r|l}
18b^3 & a^3 - 15b^4 & a^2 + 6ab^2 \\
-15b & +12b^2 & \\
-30b^4 & +\quad 9b^3 & \\
+25b^2 & -10 &
\end{array}\right.$$

$$\begin{array}{r|r|l}
18b^5 & a^3 + 12b^2 & a^2 \\
-30b^4 & -10 & \\
-15b & & \\
+25b^2 & & \\ \hline
\end{array}$$

$$\text{2d Rem.}\left\{\begin{array}{r|l}
-15b^4 & a^2 + 6ab^2 \\
+\quad 9b^2 &
\end{array}\right.$$

$$\begin{array}{r|l}
9b^2 & a^2 + 6ab^2 \\
-15b^4 & \\ \hline
0 & \quad 0
\end{array}$$

13. Divide $12x^4 - 192$ by $3x - 6$.
Quo. $4x^3 + 8x^2 + 16x + 32$.

14. Divide $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ by $a^2 - 2ab + b^2$.
Quo. $a^3 - 3a^2b + 3ab^2 - b^3$.

15. Divide $b^6 - 3b^4x^2 + 3b^2x^4 - x^6$ by $b^3 - 3b^2x + 3bx^2 - x^3$.
Quo. $b^3 + 3b^2x + 3bx^2 + x^3$.

16. Divide $a^3 + 5a^2x + 5ax^2 + x^3$ by $a + x$.
Quo. $a^2 + 4ax + x^2$.

17. Divide 1 by $1 - a$.

$$\begin{array}{llllll}
1 & \left\{ \begin{array}{l} 1 - a \\ \hline 1 + a + a^2 + a^3 + a^4 + \&c. \end{array} \right. & & & & \\
1 - a & & & & & \\
\hline
 & a & & & & \\
 & a - a^2 & & & & \\
\cline{2-3}
 & & a^2 & & & \\
 & & a^2 - a^3 & & & \\
\cline{3-4}
 & & & a^3 & & \\
 & & & a^3 - a^4 & & \\
\cline{4-5}
 & & & & a^4 & \\
 & & & & a^4 - a^5 & \\
\cline{5-6}
 & & & & & a^5
\end{array}$$

18. Divide 1 by $1 + a$.
Quo. $1 - a + a^2 - a^3 + a^4 - \&c.$

19. Divide $25x^6 - x^4 - 2x^3 - 8x^2$ by $5x^3 - 4x^2$.
Quo. $5x^3 + 4x^2 + 3x + 2$.

20. Divide $a^4 + 8a^3x + 24a^2x^2 + 32ax^3 + 16x^4$ by $a + 2x$.
Quo. $a^3 + 6a^2x + 12ax^2 + 8x^3$.

21. Divide $x^4 - \frac{5}{4}x + \frac{11}{8}x^2 - \frac{1}{2}x$ by $x^2 - \frac{1}{2}x$.
Quo. $x^2 - \frac{3}{4}x + 1$.

(17.) We shall now proceed to show some of the applications of algebra, in investigating the properties of numbers.

1. Suppose we have two numbers, the sum of which is 48, and their difference 10, and it is required to find those numbers.

By making use of the letter x, to represent an unknown quantity, we shall have the following solution:—

Let $x =$ the less number, then since the difference is 10, $x + 10$ will be the greater; and adding these numbers together, they will evidently equal the sum, so that we shall have, $2x + 10 = 48$; and if we take away 10 from each of these equal parts, the remainders will be equal, and we shall have, $2x = 48 - 10$, and hence, $x = \frac{48 - 10}{2} = 24 - 5 = 19$.

If we make $x =$ the greater, then $x - 10$ will equal the less, and we shall have, $2x - 10 = 48$; and adding 10 to each of the equal quantities, we have,

$$2x = 48 + 10,$$

$$\text{and } x = \frac{48 + 10}{2} = 24 + 5 = 29.$$

By inspecting these results, it will appear, that the less number is equal to half the sum, *minus* half the difference; and the greater equal to half the sum, *plus* half the difference.

We might infer that this property belonged to all numbers, but strictly we have proved it to belong only to the numbers under consideration. Now, to show that this property is universal, we may take a for the sum of two numbers, and b for their difference, and performing the same operations, we shall have, first,

$$2x + b = a$$

$$2x = a - b,$$

$$\text{and } x = \frac{a - b}{2}, \text{ or } \frac{a}{2} - \frac{b}{2}.$$

Taking x for the greater number,

$$2x - b = a$$

$$2x = a + b,$$

$$\text{and } x = \frac{a + b}{2}, \text{ or } \frac{a}{2} + \frac{b}{2}.$$

It is manifest, that whatever values be assigned to a and b, the form of the result will not be altered; and hence, the property demonstrated belongs to all numbers.

2. If we take any two numbers, for example, 10 and 16, and multiply their sum by their difference, the product will

be equal to the difference of their squares, as may be found by trial.

$$\begin{array}{r} 10+16=26 \\ 16-10=6 \\ \hline 156 \end{array}$$ product of the sum and difference.

16^2-10^2, or $256-100=156$, difference of the squares.

To prove this property general, let a be one number, and b another, $a+b$ is their sum, and $a-b$ their difference. Multiplying these together, we have

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad\;\, -ab-b^2 \\ \hline a^2 \qquad\;\; -b^2 \end{array}$$

And, as any numbers whatever may be substituted in the place of a and b, it follows, that this property is general. This property may be thus enunciated:—

The product of the sum and difference of two quantities is equal to the difference of their squares.

Having demonstrated this property, we can make use of it to shorten the operation of multiplication or division.

Let it be required to multiply $5ab-2a^2c$ by $5ab+2a^2c$. Since the multiplicand is the difference of two quantities, and the multiplier the sum of the same quantities, their product will be equal to the difference of their squares; and hence, without performing the multiplication, we can come directly at the result by taking the difference of the squares of the two terms.*

Wherefore, $(5ab-2ac)\times(5ab+2ac)=25a^2b^2-4a^2c^2$.

We can also reverse this operation. Let it be required to divide $64a^2b^4-36a^4b^2$ by $8ab^2-6a^2b$. Since the divisor is the difference of two quantities, and the dividend the difference of the squares of the same quantities, the quotient must be the sum of these quantities; and hence the quotient may be

---

* The square of a monomial is found by squaring the coefficient, and doubling the exponents of the letters; and the square root is found by extracting the root of the coefficient, and taking half the exponent of each letter.

By making use of the letter x, to represent an unknown quantity, we shall have the following solution:—

Let $x =$ the less number, then since the difference is 10, $x + 10$ will be the greater; and adding these numbers together, they will evidently equal the sum, so that we shall have, $2x + 10 = 48$; and if we take away 10 from each of these equal parts, the remainders will be equal, and we shall have, $2x = 48 - 10$, and hence, $x = \frac{48 - 10}{2} = 24 - 5 = 19$.

If we make $x =$ the greater, then $x - 10$ will equal the less, and we shall have, $2x - 10 = 48$; and adding 10 to each of the equal quantities, we have,

$$2x = 48 + 10,$$

$$\text{and } x = \frac{48 + 10}{2} = 24 + 5 = 29.$$

By inspecting these results, it will appear, that the less number is equal to half the sum, *minus* half the difference; and the greater equal to half the sum, *plus* half the difference.

We might infer that this property belonged to all numbers, but strictly we have proved it to belong only to the numbers under consideration. Now, to show that this property is universal, we may take a for the sum of two numbers, and b for their difference, and performing the same operations, we shall have, first, $2x + b = a$

$$2x = a - b,$$

$$\text{and } x = \frac{a - b}{2}, \text{ or } \frac{a}{2} - \frac{b}{2}.$$

Taking x for the greater number,

$$2x - b = a$$

$$2x = a + b,$$

$$\text{and } x = \frac{a + b}{2}, \text{ or } \frac{a}{2} + \frac{b}{2}.$$

It is manifest, that whatever values be assigned to a and b, the form of the result will not be altered; and hence, the property demonstrated belongs to all numbers.

2. If we take any two numbers, for example, 10 and 16, and multiply their sum by their difference, the product will

and $36a^2x^4 + 60a^3x^3 + 25a^4x^2 = (6ax^2 + 5a^2x)^2$, or

$$(6ax^2 + 5a^2x) \times (6ax^2 + 5a^2x).$$

2. *The square of the difference of two quantities is equal to the sum of their squares,* minus *twice their product.*

Let a be the greater of two quantities, and b the less, then $a - b$ will be the difference, and

$$(a - b)^2 = (a - b) \times (a - b) = a^2 - 2ab + b^2.$$

The same applications may be made of this property as of the preceding one, with this difference only, that the sign connecting the terms of the root is always —. The sign of the second term of the product will always determine, whether it is the square of the sum or difference of two quantities, for in the square of the sum it is +, and in the other —. Thus,

$$(5a - 3b)^2 = 25a^2 - 30ab + 9b^2,$$

$$(16a^2x^3 - 10a^3x)^2 = 256a^4x^6 - 320a^5x^4 + 100a^6x^2,$$

$$\text{and } \frac{16a^2b^2 - 24ab^2 + 9b^2}{4ab - 3b} = 4ab - 3b.$$

(18.) We can make use of the preceding results promiscuously, when we wish to find the factors of which a quantity is composed. For example; if it be required to decompose the quantity $8a^3 - 8ab^2$, into as many factors as possible; we first find that $8a$ will divide both terms; we have then the factors $8a$, and $a^2 - b^2$, the latter of which is equal to $(a+b)\ (a-b)$. We have therefore decomposed the quantity into three factors, $8a$, $a + b$ and $a - b$, and it may be written $8a\,(a + b) \times (a - b)$. In like manner,

$$7a^4x^2 - 28a^2x^4 = 7a^2(a^2x^2 - 4x^4) = 7a^2(ax + 2x^2)\ (ax - 2x^2),$$

$$\text{or } 7a^2x^2(a + 2x)\ (a - 2x.)$$

EXAMPLES.

1. Decompose $6a^4b - 12a^2b^3 + b^5$ into as many factors as possible,

$$6a^4b - 12a^2b^3 + 6b^5 = 6b\,(a^4 - 2a^2b^2 + b^4)$$

$$= 6b\,(a^2 - b^2)\ (a^2 - b^2) = 6b\,(a + b)\ (a - b)\ (a + b)\ (a - b).$$

2. Decompose $5a^2x - 10abx + 5b^2x$ into three factors.

Ans. $5x(a - b)\ (a - b)$.

3. Find the factors of $25a^3 - 30a^2b + 9ab^2$

Ans. $a(5a - 3b)(5a - 3b)$.

4. Find the factors of $9a^4 + 12a^2b + 4b^2$.

Ans. $(3a^2 + 2b)(3a^2 + 2b)$.

5. Divide $196a^5b^3c^2 + 336a^3b^3c^3 + 144ab^3c^4$ into its factors.

Ans. $4ab(7a^2bc + 6bc^2)(7a^2bc + 6bc^2)$.

or, $4ab^3(7a^2c + 6c^2)(7a^2c + 6c^2)$, or $4ab^3c^2(7a^2 + 6c)(7a^2 + 6c)$.

The reasoning applied to binomials may be extended to polynomials, so as often to shorten the process of multiplication. For example; if we wish to multiply $a^2 - ab + b^2 - c$ by $a^2 - ab - b^2 + c$, we observe that the first of these quantities is the sum of $a^2 - ab$, and $b^2 - c$, and the second is the difference of these same quantities, since $a^2 - ab - (b^2 - c) = a^2 - ab - b^2 + c$.

We shall have, therefore, their product equal to the difference of their squares, that is $= (a^2 - ab)^2 - (b^2 - c)^2 = a^4 - 2a^3b + a^2b^2 - (b^4 - 2b^2c + c^2)$, or $a^4 - 2a^3b + a^2b^2 - b^4 + 2b^2c - c^2$.

If it were required to multiply $2a + 3ab + 5b$ by itself, we should have, by considering it the sum of the two quantities $2a + 3ab$ and $5b$, $(2a + 3ab)^2 + 2 \times 5b(2a + 3ab) + (5b)^2$ for the product; or, performing the operations indicated,

$$4a^2 + 12a^2b + 9a^2b^2 + 20ab + 30ab^2 + 25b^2.$$

Again, $(3a^2 - 2ab - 4bc)^2 = (3a^2 - 2ab)^2 - 2 \times 4bc(3a^2 - 2ab) + (4bc)^2 = 9a^4 - 12a^3b + 4a^2b^2 - 24a^2bc + 16ab^2c + 16b^2c^2$.

(19.) By applying the principles demonstrated in the preceding pages, we may change the form of polynomials by decomposing them into factors, and then uniting these factors in a different way. Thus,

$$2a^5c - 4a^3b^2c + 2ab^4c = 2ac(a^4 - 2a^2b^2 + b^4)$$
$$= 2ac(a^2 - b^2)(a^2 - b^2) = 2ac(a + b)(a - b)(a + b)(a - b)$$
$$= 2ac(a + b)(a + b)\ a - b\ (a - b) = 2ac(a + b)^2(a - b)^2$$
$$= 2ac(a^2 + 2ab + b^2)(a^2 - 2ab + b^2).$$

For another example, take the quantity

$$a^2 + b^2 - c^2 + 2ab.$$

This may be written,

$$a^2 + 2ab + b^2 - c^2.$$

The first three terms being the square of $a+b$, we change the expression to

$$(a+b)^2-c^2.$$

But this last expression is the difference of two squares; and hence we may substitute for it, the product of the sum and difference, and we obtain,

$$(a+b)^2-c^2=(a+b+c)(a+b-c).$$

For a third example of the mutations which an algebraic quantity may be made to undergo, take the following, which becomes successively,

$$a^2-b^2-x^2-2bx,$$
$$(a-b)(a+b)-x^2-2bx,$$
$$(a-x)(a+x)-b^2-2bx,$$
$$a^2-(x^2+2bx+b^2),$$
$$a^2-(x+b)^2,$$
$$(a-x-b)(a+x+b).$$

In like manner, $a^2-b^2-x^2+2bx$, may be written,

$$a^2-(b^2-2bx+x^2),$$

or, $a^2-(b-x)^2,$

or, $(a-b+x)(a+b-x).$

FRACTIONS.

(20.) Since algebraic fractions represent unexecuted divisions, and are in all respects similar to arithmetical fractions, the same rules will apply to both.

REDUCTION OF FRACTIONS.

To reduce fractions to their lowest terms.

RULE.

Divide both terms of the fraction by any quantity that will divide them without remainder. Divide these quotients again, in like manner, till it appears that no quantity can be found which will exactly divide both; or, which is the same thing, suppress all the factors common to both terms of the fraction.

EXAMPLES.

1. Reduce $\frac{14a^2b^3c}{21abc^2}$ to its lowest terms.

Here $7abc$ is the common divisor, and dividing according to the rule, gives $\frac{2ab^2}{3c}$, the fraction reduced.

2. Reduce $\frac{33a^2x^2}{55a^2x^3}$. Ans. $\frac{3}{5x}$.

3. Reduce $\frac{116a^5x^2y}{68a^3xy^2}$. Ans. $\frac{29a^2x}{17y}$.

4. Reduce $\frac{51a^3b - 63a^2b^2}{36a^4b^2 - 9ab}$. Ans. $\frac{17a^2 - 21ab}{12a^3b - 9}$.

5. Reduce $\frac{14a^2x - 28a^3x^2 + 7a^4x^3}{49a^2x^2 - 56a^3x}$.

Ans. $\frac{2 - 4ax + a^2x^2}{7x - 8a}$.

6. Reduce $\frac{a - b}{a^2 - b^2}$. Ans. $\frac{1}{a + b}$.

7. Reduce $\frac{10a^3x + 20a^2x^2 + 10ax^3}{a^3 - ax^2}$.

Decomposing each term of this fraction into its factors, we have,

$$\frac{10ax(a^2 + 2ax + x^2)}{a(a^2 - x^2)} = \frac{10ax(a + x)(a + x)}{a(a - x)(a + x)};$$

and suppressing factors common to both,

$\frac{10x(a + x)}{a - x}$, or $\frac{10ax + 10x^2}{a - x}$, the fraction in its lowest terms.

8. Reduce $\frac{5a^5b^2 - 20ab^6}{a^4b^2 + 2ab^4}$.

$$\frac{5a^5b^2 - 20ab^6}{a^4b^2 + 2a^2b^4} = \frac{5ab^2(a^4 - 4b^4)}{a^2b^2(a^2 + 2b^2)} = \frac{5(a^2 - 2b^2)(a^2 + 2b^2)}{a(a^2 + 2b^2)}$$

$$= \frac{5(a^2 - 2b)}{a}, \text{ the answer.}$$

9. Reduce $\frac{8a^3 - 32ab^2}{2a^4 - 8a^3b + 8a^2b^2}$.

Ans. $\frac{4(a + 2b)}{a(a - 2b)}$, or $\frac{4a + 8b}{a^2 - 2ab}$.

10. Reduce $\frac{a^4+2a^2cx^2+c^2x^4}{5a^3cx+5ac^2x^3}$. Ans. $\frac{a^2+cx^2}{5acx}$.

It is sometimes the case, that the terms of the fraction are so complicated, that it is not possible to reduce them by the preceding rule, and it then becomes necessary to find their greatest common divisor, and divide each term of the fraction by it, in order to effect the reduction.

To find the greatest common divisor of two polynomials.

RULE.

(21.) 1st. *Arrange the polynomials with reference to the powers of the same letter.*

2d. *Divide the polynomial, which has the highest exponent of this letter, by the other; and if there be a remainder, divide the divisor by this remainder, and continue the operation in the same manner, always dividing the last divisor by the last remainder, till the division can be executed without remainder; the last divisor is the greatest common divisor of the two polynomials.*

NOTE.—If, in the course of the operation, it happens at any time, that any factor is contained in all the terms of one polynomial, and not in all the terms of the other; this factor may be expunged from the polynomial containing it; for, since it divides one of the polynomials, and will not divide the other, it cannot be a factor of their common divisor.

EXAMPLES.

1. Find the greatest common divisor of a^5-ax^4, and $a^3+a^2x-ax^2-x^3$.

In this example, a^5-ax^4 contains a in all its terms, and the other does not: a therefore cannot be a factor of the common divisor; for if it were, it must divide $a^3+a^2x-ax^2-x^3$; we can therefore expunge it from the first; and the problem is, to find the greatest common divisor of a^4-x^4, and $a^3+a^2x-ax^2-x^3$.

$$\begin{array}{l|l}
\text{1st dividend.} & \text{1st divisor.} \\
a^4 - x^4 & a^3 + a^2x - ax^2 - x^3 \\ \cline{2-2}
 & a - x \\
a^4 + a^3x - a^2x^2 - ax^3 & \\ \cline{1-1}
-a^3x + a^2x^2 + ax^3 - x^4 & \\
-a^3x - a^2x^2 + ax^3 + x^4 & \\ \cline{1-1}
2a^2x^2 - 2x^4 = 2x^2(a^2 - x^2) &
\end{array}$$

The division can be carried no farther, since the first term of the divisor is not contained in any term of the dividend; we therefore take the divisor for the dividend, and the remainder for a new divisor, expunging from it the factor $2x^2$, which is not found in all the terms of the new dividend.

$$\begin{array}{l|l}
\text{2d dividend.} & \text{2d divisor.} \\
a^3 + a^2x - ax^2 - x^3 & a^2 - x^2 \\ \cline{2-2}
a^3 - ax^2 & a + x \\ \cline{1-1}
a^2x - x^3 & \\
a^2x - x^3 & \\ \cline{1-1}
0 \quad 0 &
\end{array}$$

$a^2 - x^2$ is therefore the greatest common divisor.

2. Find the greatest common divisor of $12a^4 - 24a^3b + 12a^2b^2$, and $8a^3b^2 - 24a^2b^3 + 24ab^4 - 8b^5$.

Expunging factors not common to both, these quantities become $a^2 - 2ab + b^2$, and $a^3 - 3a^2b + 3ab^2 - b^3$.

$$\begin{array}{l|l}
\text{Dividend.} & \text{Divisor.} \\
a^3 - 3a^2b + 3ab^2 - b^3 & a^2 - 2ab + b^2 \\ \cline{2-2}
a^3 - 2a^2b + ab^2 & a - b^2 \\ \cline{1-1}
-a^2b^2 + 2ab^2 - b^3 & \\
-a^2b + 2ab^2 - b^3 & \\ \cline{1-1}
0 \quad 0 \quad 0 &
\end{array}$$

$a^2 - 2ab + b^2$ is therefore the greatest common divisor.

NOTE 2.—If the letter with respect to which the quantities are arranged, have the highest exponent in the dividend, and the *coefficient* of the first term of the dividend be not divisible by the *coefficient* of the first term of the divisor, the dividend must be multiplied by such a number as will make this coefficient divisible. This will not affect the correctness of the result, for the factor thus introduced into the dividend, and not into the divisor, can form no part of the *greatest common divisor.*

3. Find the greatest common divisor of $3a^4 - 5a^3b + 5a^2b^2 - 5ab^3 + 2b^4$, and $6a^3 + 8a^2b - 11ab^2 + 2b^3$.

1st division.

$$\begin{array}{l|l}
2\times 3a^4 - 5a^3b + 5a^2b^2 - 5ab^3 + 2b^4 & 6a^3+8a^2b-11ab^2+2b^3 \\ \hline
6a^4 - 10a^3b + 10a^2b^2 - 10ab^3 + 4b^4 & a - 3b \\
6a^4 + 8a^3b - 11a^2b^2 + 2ab^3 & \\ \hline
\quad - 18a^3b + 21a^2b^2 - 12ab^3 + 4b^4 & \\
\quad - 18a^3b - 24a^2b^2 + 33ab^3 - 6b^4 & \\ \hline
\qquad 45a^2b^2 - 45ab^3 + 10b^4, \text{ or} & \\
\qquad 5b^2(9a^2 - 9ab + 2b^2) &
\end{array}$$

2d division.

$$\begin{array}{l|l}
3\times 6a^3 + 8a^2b - 11ab^2 + 2b^3 & 9a^2 - 9ab + 2b^2 \\ \hline
18a^3 + 24a^2b - 33ab^2 + 6b^3 & 2a + 14b \\
18a^3 - 18a^2b + 4ab^2 & \\ \hline
3\times 42a^2b - 37ab^2 + 6b^3 & \\ \hline
126a^2b - 111ab^2 + 18b^3 & \\
126a^2b - 126ab^2 + 28b^3 & \\ \hline
\qquad 15ab^2 - 10b^3, & \\
\text{or } 5b^2(3a - 2b). &
\end{array}$$

3d division.

$$\begin{array}{l|l}
9a^2 - 9ab + 2b^2 & 3a - 2b \\ \hline
9a^2 - 6ab & 3a - b \\ \hline
- 3ab + 2b^2 & \\
- 3ab + 2b^2 & \\ \hline
\end{array}$$

$3a - 2b$, the last divisor, is the greatest common divisor required.

We now propose to find the common divisor of these same polynomials by arranging them with reference to the powers *of b.*

1st division.

$$\begin{array}{l|l}
2b^4 - 5ab^3 + 5a^2b^2 - 5a^3b + 3a^4 & 2b^3 - 11ab^2 + 8a^2b + 6a^3 \\
\underline{2b^4 - 11ab^3 + 8a^2b^2 + 6a^3b} & \overline{b + 3a} \\
\quad 6ab^3 - 3a^2b^2 - 11a^3b + 3a^4 & \\
\quad \underline{6ab^3 - 33a^2b^2 + 24a^3b + 18a^4} & \\
\qquad 30a^2b^2 - 35a^3b - 15a^4, & \\
\text{or} \quad 5a^2(6b^2 - 7ab - 3a^2). &
\end{array}$$

2d division.

$$\begin{array}{l|l}
3 \times \underline{2b^3 - 11ab^2 + 8a^2b + 6a^3} & 6b^2 - 7ab - 3a^2 \\
\quad 6b^3 - 33ab^2 + 24a^2b + 18a^3 & \overline{b - 13a} \\
\quad \underline{6b^3 - 7ab^2 - 3a^2b} & \\
3 \times \underline{-26ab^2 + 27a^2b + 18a^3} & \\
\quad -78ab^2 + 81a^2b + 54a^3 & \\
\quad \underline{-78ab^2 + 91a^2b + 39a^3} & \\
\qquad -10a^2b + 15a^3 & \\
\text{or} \qquad 5a^2(-2b + 3a) &
\end{array}$$

3d division.

$$\begin{array}{l|l}
6b^2 - 7ab - 3a^2 & \underline{-2b + 3a}, \text{ or } 3a - 2b, \text{ the gr'st com. div.} \\
\underline{6b^2 - 9ab} & -3b - a \\
\quad +2ab - 3a^2 & \\
\quad \underline{+2ab - 3a^2} &
\end{array}$$

4. Find the greatest common divisor of $a^3 - ab^2$ and $a^2 + 2ab + b^2$. Ans. $a + b$.

5. Find the greatest common divisor of $\frac{a^5 - a^3b^2}{a^4 - b^4}$.

Ans. $a^2 - b^2$.

6. Reduce $\frac{a^3x + 2a^2x^2 + 2ax^3 + x^4}{5a^5 + 10a^4x + 5a^3x^2}$ to its lowest terms.

Ans. $\frac{a^2x + ax^2 + x^3}{5a^4 + 5a^3x}$.

7. Reduce $\frac{2x^3 - 16x - 6}{3x^3 - 24x - 9}$ to its lowest terms.

Ans. $\frac{2}{3}$.

8. Reduce $\frac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$ to its lowest terms.

Ans. $\frac{3a-x}{3a+x}$.

To reduce a mixed quantity to an improper fraction.

RULE.

(22.) *Multiply the quantity to which the fraction is annexed by the denominator of the fraction, and connect the product, by its proper sign, with the numerator.*

EXAMPLES.

1. Reduce $3a+\frac{2x}{5a^2}$ to an improper fraction.

$$(3a\times 5a^2)+2x=15a^3+2x.$$

Hence, $\frac{15a^3+2x}{5a^2}$ is the fraction required.

2. Reduce $5x-\frac{4bx^2}{3bx}$ to the form of a fraction.

Ans. $\frac{11bx^2}{3bx}$ or $\frac{11x}{3}$.

3. Reduce $5x-\frac{2x-3}{7}$ to the form of a fraction.

The sign — before the fractional part, shows that the whole fraction is to be subtracted from $5x$; and hence, the signs of the numerator are to be changed, when we connect it with the product of $5x$ by the denominator.

Wherefore, $5x-\frac{2x-3}{7}=\frac{35x-2x+3}{7}=\frac{33x+3}{7}$.

4. Reduce $a-x-\frac{a^2-ax}{x}$ to the form of a fraction.

Ans. $\frac{2ax-a^2-x^2}{x}$.

5. Reduce $3a^2-x+\frac{3a^4+ax^2}{a^2-ax}$ to the form of a fraction.

Ans. $\frac{6a^3-ax-3a^2x+2x^2}{a-x}$.

6. Reduce $2a+b-\frac{3a^2-b^2}{2a-b}$ to the form of a fraction.

Ans. $\frac{a^2}{2a-b}$.

7. Reduce $16-\frac{a+8}{5}$ to the form of a fraction.

Ans. $\frac{72-a}{5}$.

8. Reduce $a^2+3ab+b^2+\frac{a^3-2a^2b+b^3}{a-b}$ to the form of a fraction.

Ans. $\frac{2a^3-2ab^2}{a+b}$.

9. Reduce $3a-9-\frac{3a^2-30}{a+3}$ to the form of a fraction.

Ans. $\frac{3}{a+3}$.

To reduce improper fractions to whole or mixed quantities.

RULE.

(23.) *Divide the numerator by the denominator, and if there be a remainder, write the divisor beneath it, and connect the fraction so formed, by its proper signs, with the quotient.*

EXAMPLES.

1. Reduce $\frac{15a^3+2x}{5a^2}$ to a mixed quantity.

Ans. $3a+\frac{2x}{5a^2}$.

2. Reduce $\frac{a^2+ab+b^2}{a}$ to a mixed quantity.

Ans. $a+b+\frac{b^2}{a}$.

3. Reduce $\frac{12a^2+4a-3c}{4a}$ to a mixed quantity.

Ans. $3a+1-\frac{3c}{4a}$.

4. Reduce $\frac{a^4 - x^4}{a^2 - ax + x^2}$ to a mixed quantity.

Ans. $a^2 + ax - \frac{ax^3 + x^4}{a^2 - ax + x^2}$.

5. Reduce $\frac{a^5x - 3ax^5}{a^3 - ax^2}$ to a mixed quantity.

Ans. $a^2x + x^3 - \frac{2ax^5}{a^3 - ax^2}$.

6. Reduce $\frac{25a^4c^2 - 20a^2bc + 4b^2}{5a^2c - 2b}$ to a mixed quantity.

Ans. $5a^2c - 2b$.

7. Reduce $\frac{9a^2b^2 + 3ab - 4c^2 - 2c}{3ab + 2c}$ to a mixed quantity.

Ans. $3ab - 2c + \frac{3ab - 2c}{3ab + 2c}$.

8. Reduce $\frac{12ax^2 + 6x^3 - 3}{6x^2}$ to a mixed quantity.

Ans. $2a + x - \frac{3}{6x^2}$.

MULTIPLICATION OF FRACTIONS.

RULE.

(24.) *Multiply the numerators together for a new numerator, and the denominators for a new denominator, and reduce the resulting fraction to its lowest terms.*

EXAMPLES.

1. Multiply $\frac{3x}{4}$ by $\frac{5}{7x^2}$.

$$\frac{3x \times 5}{4 \times 7x^2} = \frac{15x}{28x^2} = \frac{15}{28x},\text{ the answer.}$$

2. Multiply $\frac{4x + 1}{3}$ by $\frac{6x}{7}$.

Ans. $\frac{24x^2 + 6x}{21} = \frac{8x^2 + 2x}{7}$.

3. Multiply $\frac{a^2 - b^2}{5b}$ by $\frac{3a^2}{a + b}$.

Ans. $\frac{a^2 - b^2 \times 3a^2}{5b \times a + b} = \frac{(a^2 - b^2)3a^2}{(a + b)5b} = \frac{(a + b)\ (a - b)3a^2}{(a + b) \times 5b}$

$= \frac{3a^3 - 3a^2b}{5b}$.

4. Multiply $\frac{3x^2 - 5x}{14}$ by $\frac{7a}{2x^3 - 3x}$.

Ans. $\frac{3ax - 5a}{4x^2 - 6}$.

5. Multiply $\frac{3x^2}{5x - 10}$ by $\frac{15x - 30}{2x}$. Ans. $\frac{9x}{2}$.

6. Multiply $\frac{8ab}{3}$ by $\frac{3}{8ab}$. Ans. 1.

7. Multiply $\frac{4a^2 - 16b^2}{a - 2b}$ by $\frac{5b}{8a^2 + 32ab + 32b^2}$.

$$\frac{(4a^2-16b^2)5b}{(a-2b)(8a^2+32ab+3b^2} = \frac{(2a-4b)(2a+4b)5b}{2(a-2b)(4a^2+16ab+16b^2)}$$

$$= \frac{(2a - 4b)(2a + 4b)5b}{(2a - 4b)(2a + 4b)(2a + 4b)} = \frac{5b}{2a + 4b}\text{; the answer.}$$

8. Multiply $\frac{a^4 - b^4}{a + b}$ by $\frac{a^2}{ab - b^2}$.

Ans. $\frac{a^4 + a^2b^2}{b}$.

9. Multiply $\frac{a^2x - x^3}{a}$ by $\frac{3a}{2ax - 2x^2}$.

Ans. $\frac{3a + 3x}{2}$.

Note.—When a fraction is to be multiplied by an entire quantity, the numerator may be multiplied, or the denominator divided by this quantity (Arith. 20); and hence, if the multiplier be the same as the denominator, the denominator becomes unity, and therefore, *to multiply a fraction by a quantity equal to its denominator, it is only necessary to expunge the denominator.*

Thus, $\frac{7ax}{8} \times 2 = \frac{14ax}{8}$ or $\frac{7ax}{4}$.

$\frac{a^2 - b^2}{2a^2} \times a = \frac{a^3 - ab^2}{2a^2}$, or $\frac{a^2 - b^2}{2a}$, and

$\frac{ax^2}{2b} \times 2b = ax^2$; also,

$\frac{13}{6a^2 - 2b} \times 6a^2 - 2b = 13.$

DIVISION OF FRACTIONS.

RULE.

(24.) *Invert the divisor, and then proceed as in multiplication.*

EXAMPLES.

1. Divide $\frac{14x^2}{9}$ by $\frac{2x}{3}$.

$$\frac{14x^2}{9}\times\frac{3}{2x}=\frac{42x^2}{18x}=\frac{7x}{3}, \text{ the answer.}$$

2. Divide $\frac{5a^2-5b^2}{2a}$ by $\frac{2a+2b}{3b}$.

$$\frac{5a^2-5b^2}{2a}\times\frac{3b}{2a+2b}=\frac{15a^2b-15b^3}{4a^2+4ab}=\frac{15b(a^2-b^2)}{4a(a+b)}$$

$$=\frac{15b(a-b)}{4a}, \text{ the answer.}$$

3. Divide $\frac{x^4-b^4}{x^2-2bx+x^2}$ by $\frac{x^2+bx}{x-b}$.

$$\frac{(x^4-b^4)(x-b)}{(x^2-2bx+x^2)(x^2+bx}=\frac{(x^2-b^2)(x^2+b^2)(x-b)}{(x-b)^2\times x(x+b)}$$

$$=\frac{(x-b)(x+b)(x^2+b^2)(x-b)}{(x-b)(x-b)\times x(x+b}=\frac{x^2+b^2}{x}$$

$$=x+\frac{b^2}{x}, \text{ the answer.}$$

4. Divide $\frac{a^3x^2}{7}$ by $\frac{4a^2}{x^2}$. Ans. $\frac{ax^4}{28}$.

5. Divide $\frac{15}{2a^2b}$ by $\frac{10ab^2}{4}$. Ans. $\frac{3}{a^3b^3}$.

6. Divide $\frac{14x-3}{5}$ by $\frac{10x-4}{25}$. Ans. $\frac{70x-15}{10x-4}$.

7. Divide $\frac{9x^2-3x}{5}$ by $\frac{x^2}{5}$. Ans. $\frac{9x-3}{x}$.

8. Divide $\frac{6x-7}{x+1}$ by $\frac{x-1}{3}$. Ans. $\frac{18x-21}{x^2-1}$.

9. Divide $\frac{6(a+x)+2}{3}$ by $\frac{4}{3(a-x)}$.

Ans. $\frac{3)a^2-x^2)+a-x}{2}$.

10. Divide 8 by $\frac{a(a-x)^2}{a^3-ax^2}+a$. Ans. $4+\frac{4x}{a}$.

11. Divide $\frac{x+x^2}{3a^2}$ by $\frac{2ax+2ax^2}{7}$. Ans. $\frac{7}{6a^3}$.

12. Divide $\frac{2a^6x-2x^7}{a^2x-2ax^2+x^3}$ by $\frac{a^2+ax+x^2}{a-x}$.

Ans. $2(a^3+x^3.)$

Note.—When a fraction is to be divided by an entire quantity, the operation may be performed either by dividing the numerator, or multiplying the denominator by this quantity. Thus,

$$\frac{3x}{7}\div 3=\frac{x}{7} \text{ or } \frac{3x}{21};\ \frac{2x}{a}\div a=\frac{2x}{a^2}, \text{and } \frac{a^3}{b}\div a-b$$
$$=\frac{a^3}{ab-b^2}.$$

To reduce fractions of different denominators to equivalent fractions having a common denominator.

RULE.

(25.) *Multiply all the denominators together for a common denominator, and each numerator into all the denominators, except its own, for new numerators*; or,

Find the least common multiple of all the denominators; this will be the common denominator. For the numerators, divide the common denominator, thus found, by each particular denominator, and multiply the quotient by the numerator.

EXAMPLES.

1. Reduce $\frac{a}{z}$ and $\frac{b}{x}$ to a common denominator.

$x\times z=xz$, the common denominator; ax and bz the numerators. The fractions reduced, are, therefore,

$$\frac{ax}{xz} \text{ and } \frac{bz}{xz}.$$

2. Reduce $\frac{a}{c}, \frac{b}{d}$ and $\frac{c}{e}$ to a common denominator.

Ans. $\frac{ade}{cde}, \frac{bce}{cde}$ and $\frac{c^2d}{cde}$.

3. Reduce $\frac{5a}{3x}, \frac{3b}{2c}$ and $4d$ to a common denominator.

Ans. $\frac{10ac}{6cx}, \frac{9bx}{6cx}$ and $\frac{24cdx}{6cx}$.

4. Reduce $\frac{5}{6}, \frac{3a}{4}$ and $2b + \frac{3a}{b}$ to a common denominator.

Ans. $\frac{20b}{24b}, \frac{18ab}{24b}$ and $\frac{48b^2 + 72a}{24b}$.

5. Reduce $\frac{1}{3}, \frac{2a^2}{4}$ and $\frac{2a^2 + b^2}{a + b}$ to a common denominator.

Ans. $\frac{4a + 4b}{12a + 12b}, \frac{6a^3 + 6a^2b}{12a + 12b}$ and $\frac{24a^2 + 12b^2}{12a + 12b}$.

6. Reduce $\frac{3b}{4a^3}, \frac{2c}{3a^2}$ and $\frac{d}{2a}$ to a common denominator.

In this example, the denominators have a common multiple, less than their continued product; and if we would reduce the fractions to their *least* common denominator, it is necessary to find this least common multiple. In algebraic fractions, this can generally be done by inspection, as in this example it is easy to see that $12a^3$ is the least common multiple of the denominators. But it may also be found by a process, analogous to that given in arithmetic (Arith. 38), for finding the least common multiple of several numbers. Thus,

$$\begin{array}{r|ccc}
2a & 4a^3, & 3a^2, & 2a \\ \cline{2-4}
a^2 & 2a^2, & 3a^2, & 1 \\ \cline{2-4}
 & 2, & 3, & 1
\end{array}$$

Therefore $2a \times a^2 \times 2 \times 3 = 12a^3$, is the common multiple.

$$\begin{array}{lcl}
\text{Com. denom.} & \text{Denominators.} & \text{Numerators.} \\
12a^3 \div 4a^3 & = 3 & \times 3b = 9b \\
12a^3 \div 3a^2 & = 4a & \times 2c = 8ac \\
12a^3 \div 2a & = 6a^2 & \times d = 6a^2d
\end{array}\left.\vphantom{\begin{array}{l}1\\1\\1\end{array}}\right\}\text{New num.}$$

The fractions reduced are $\frac{9b}{12a^3}, \frac{8ac}{12a^3}$ and $\frac{6a^2d}{12a^3}$.

7. Reduce $\frac{a}{x^2y}$, $\frac{a^2}{y}$, $\frac{b}{x}$ and $\frac{c}{y^2}$ to a common denominator.

Ans. $\frac{ay}{x^2y^2}$, $\frac{a^2x^2y}{x^2y^2}$, $\frac{bxy^2}{x^2y^2}$ and $\frac{cx^2}{x^2y^2}$.

8. Reduce $\frac{a - x}{a^3x + ax^3}$ and $\frac{a + x}{ax}$ to a common denominator.

Ans. $\frac{a - x}{a^3x + ax^3}$ and $\frac{a^3 + a^2x + ax^2 + x^3}{a^3x + ax^3}$.

9. Reduce $\frac{8}{2a}$, $\frac{5}{4a^2}$ and $\frac{3}{8a^3}$ to a common denominator.

Ans. $\frac{32a^2}{8a^3}$, $\frac{10a}{8a^3}$ and $\frac{3}{8a^3}$.

10. Reduce $\frac{a^2 + b^2}{2a}$, $\frac{3b}{a - b}$ and $\frac{6a^2}{2a^2 + 2ab}$ to a common denominator.

Ans. $\frac{a^4 - b^4}{2a^3 - 2ab}$, $\frac{6a^2b + 6ab^2}{2a^3 - 2ab^2}$ and $\frac{6a^3 + 6a^2b}{2a^3 - 2ab^2}$.

ADDITION OF FRACTIONS.

RULE.

(26.) *Reduce the fractions to a common denominator; and the sum of the numerators, written over the common denominator, will be the sum of the fractions.*

EXAMPLES.

1. Add $\frac{3x}{5}$, $\frac{2x}{7}$ and $\frac{x}{3}$ together.

These fractions, reduced to a common denominator, become, respectively,

$$\frac{63x}{105}, \frac{30x}{105} \text{ and } \frac{35x}{105},$$

and their sum is

$$\frac{63x + 30x + 35x}{105} = \frac{128x}{105}.$$

2. Add $\frac{a}{b}$, $\frac{2a}{3b}$ and $\frac{5b}{4a}$ together.

Ans. $\frac{20a^2 + 15b^2}{12ab}$.

3. Add $\frac{3a^2}{2b}$, $\frac{2a}{5}$ and $\frac{3b}{7a}$ together.

Ans. $\frac{105a^3 - 28a^2b + 30b^2}{70ab}$.

4. Add $\frac{2x+1}{3}$, $\frac{4x+2}{5}$ and $\frac{x}{7}$ together.

Ans. $\frac{169x+77}{105}$.

4. Add $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$ together.

Ans. $\frac{2a^2+2b^2}{a^2-b^2}$.

6. Add $2a + \frac{a+3}{5}$ and $4a + \frac{2a-5}{4}$ together.

Ans. $6a + \frac{14a-13}{20}$.

7. Add $a - \frac{8x^2}{b}$ and $b + \frac{2ax}{c}$ together.

Ans. $a + b + \frac{2abx - 8cx^2}{bc}$.

8. Add $\frac{12a^2}{16b^4}$ and $\frac{9a}{8b^2}$ together.

Ans. $\frac{12a^2+18ab^2}{16b^4} = \frac{6a^2+9ab^2}{8b^4}$.

9. Add $\frac{1263}{27}$ and $\frac{65}{9}$ together.

Ans. $\frac{1263+195}{27} = \frac{1458}{27} = 54$.

10. Add together $\frac{125}{4}$ and $\frac{144}{16}$. Ans. $\frac{644}{16}$.

11. Add together $4x$, $\frac{7x}{9}$ and $2 + \frac{x}{5}$.

Ans. $\frac{224x+90}{45}$ or $2 + 4x + \frac{44x}{45}$.

SUBTRACTION OF FRACTIONS.

RULE.

(27.) *Reduce the fractions to a common denominator, and subtract one numerator from the other.*

EXAMPLES.

1. From $\frac{x+a}{b}$ subtract $\frac{c}{d}$.

The fractions reduced to a common denominator are,

$$\frac{dx+ad}{bd} \text{ and } \frac{bc}{bd};$$

and subtracting the numerator of the latter, gives for the difference, $\frac{dx+ad-bc}{bd}$.

2. From $3x+\frac{x}{b}$ subtract $x-\frac{x-a}{c}$.

Reducing the mixed quantities, we have

$$\frac{3bx+x}{b} \text{ and } \frac{cx-x+a}{c},$$

which reduced to a common denominator become

$$\frac{3bcx+cx}{bc} \text{ and } \frac{bcx-bx+ab}{bc}.$$

Subtracting, according to the rule, we have for the difference,

$$\frac{3bcx+cx-bcx+bx-ab}{bc}=\frac{2bcx+cx+bx-ab}{bc},$$

$$\text{or, } 2x+\frac{cx+bx-ab}{bc}.$$

3. From $\frac{3x}{7}$ take $\frac{2x}{9}$. Ans. $\frac{13x}{63}$.

4. From $\frac{2a-b}{4c}$ subtract $\frac{3a-4b}{3b}$.

Ans. $\frac{6ab-3b^2-12ac+16bc}{12bc}$.

5. Subtract $\frac{3x+1}{x+1}$ from $\frac{4x}{5}$.

Ans. $\frac{4x^2-11x-5}{5x+5}$.

6. Subtract $\frac{4x}{5}$ from $\frac{9x}{10}$. Ans. $\frac{x}{10}$.

7. Subtract $\frac{2x-3}{3x}$ from $\frac{4x+2}{3}$. Ans. $\frac{4x^2+3}{3x}$.

8. Subtract $\frac{1}{a+b}$ from $\frac{1}{a-b}$.

Ans. $\frac{-2b}{a^2-b^2}$, or $\frac{2b}{b^2-a^2}$.

9. Subtract $\frac{x+3}{x}$ from $\frac{x}{x-3}$. Ans. $\frac{9}{x^2-3x}$.

10. Subtract $\frac{ax-x^2}{ax+x^2}$ from $\frac{ax+x^2}{ax-x^2}$.

Ans. $\frac{4ax^3}{a^2x^2-x^4}=\frac{4ax}{a^2-x^2}$.

11. Subtract $\frac{54}{3}$ from $\frac{956}{27}$. Ans. $\frac{470}{27}$.

12. Subtract $\frac{a^2b^2}{2a+b^2}$ from $\frac{2a^4b^2}{4a^3-ab^4}$.

Ans. $\frac{a^3b^4}{4a^3-ab^4}$.

13. Subtract $\frac{13a^2}{3b}$ from $\frac{92a^3b}{12ab^2}$.

Ans. $\frac{40a^3b}{12ab^2}=\frac{10a^2}{3b}$.

CHAPTER II.

EQUATIONS.

(28.) An *equation* is the algebraic expression of equality between two quantities; $x+12=4x-18$, $ax+ab-c=2a^2$ and $\frac{5x^2}{3}+2x=\frac{20}{3}+14$ are equations.

The quantities on each side of the sign of equality are called *members*; that on the left is called the *first* member, and the other the *second* member.

The quantities which compose a member are called *terms*, when they are connected together by the signs of addition and subtraction. Thus, in the equation $ax+b=3a-c+d$, the first member has two *terms*, ax and b, and the second member three *terms*, $3a$, c, and d. But in the equation $(a-b)(2b+x)=4a^2$, $(a+b)(2b+x)$ is only one term; the quantities of which it is composed being connected by the sign of multiplication.

In every equation there is supposed to be at least one *unknown quantity* expressed by a letter, and the solution of an equation consists in finding the value of this unknown quantity.

An equation which has all its known quantities expressed by numbers, as $5x+4=11x-20$ is called a *numerical* equation; one which has its known quantities expressed by letters, as $ab-bx+a^2=3a^3-3bx$, is called a *literal* equation.

Equations are of the first, second, third or fourth degree, according as the unknown quantity which they contain is of the first, second, third or fourth power.

$ax+b=3ax$ is an equation of the *first degree*, or *simple equation.*

$ax^2+bx=3ab$ is an equation of the *second degree* or *quadratic equation.*

$ax^3+bx^2+cx=2a^4b$ is an equation of the *third degree.*

$ax^4+bx^3+cx^2+dx=2ab^5$ is an equation of the *fourth degree.*

EQUATIONS OF THE FIRST DEGREE.

(29.) In order to find the value of an unknown quantity contained in an equation of the first degree, it is only necessary so to change the *form* of the equation, without destroying the equality of the members, that the unknown quantity may stand alone on one side of the sign of equality, and the quantities on the other side be all known quantities. Thus, in the equation $x = \frac{a-b}{c}$; a, b, and c being known quantities, the value of x is determined.

To effect the changes necessary to bring the equation to this form, it is important to observe, that we may, without destroying the equation, *add* the same quantity to, or *subtract* it from, its two members; or *multiply* or *divide* its members by the same quantity. This depends upon the following AXIOMS, which are of constant use in the resolution of equations:—

1. If equal quantities be added to equal quantities, the sums will be equal.
2. If equal quantities be subtracted from equal quantities, the remainders will be equal.
3. If equal quantities be multiplied by equal quantities, the products will be equal.
4. If equal quantities be divided by equal quantities, the quotients will be equal.

In an equation, the unknown quantity may be connected with known quantities, by addition, by subtraction, by multiplication, by division, or by all these combined; and in order to disengage it, the opposite arithmetical operation must be employed.

In the equation $x+5=9$, the unknown quantity x, is connected with the known quantity 5, by addition; and to find the value of x, we subtract 5 from each member of the equation, and have $x=9-5=4$.

In the equation $x-2=13$, the unknown quantity is connected with the known quantity 2, by subtraction; and to find the value of x, we add 2 to both members, and the equation becomes $x-2+2=13+2$, or $x=13+2=15$.

In the equation $3x=27$, the unknown quantity is connected with the known quantity, 3, by multiplication; and if we divide both members by 3, we shall have $x=9$.

In the equation $\frac{x}{4} = 3$, the unknown quantity is connected with the known quantity, 4, by division; and if we multiply both numbers by 4, the equation becomes $x = 12$. Finally, in the equation

$$\frac{x}{2} + 8 = 23 - 2x,$$

the unknown quantity is connected with known quantities, by addition, subtraction, multiplication, and division; and to find the value of x, we first multiply both numbers by 2, and it becomes $x + 16 = 46 - 4x$; then we add $4x$ to each member, and have $5x + 16 = 46 - 4x + 4x$; and since $-4x$ and $+4x$ destroy each other, $5x + 16 = 46$; next we subtract 16 from each member, and have $5x = 30$; lastly, we divide both numbers by 5, and we obtain $x = 6$.

A combination of the preceding principles, gives the following practical rules for the resolution of equations.

Rule I.

(30.) *Any term may be transposed from one member of an equation to the other, by changing its sign.*

Take the equation $5x - 8 = 16 + x$.

Adding 8 to each member, the equation becomes

$$5x - 8 + 8 = 16 + x + 8;$$

and since, in the first member, $-8 + 8 = 0$, we have

$$5x = 16 + x + 8.$$

Subtracting x from each member, we have

$$5x - x = 16 + x - x + 8,$$

or, since in the second member, $+x - x = 0$,

$$5x - x = 16 + 8,$$

an equation differing from the first, only in having x and 8 in different members, with a change of sign.

In like manner, the literal equation $ax + ab = ac - bx$, becomes, by subtracting ab from, and adding bx to, both members,

$$ax + ab - ab + bx = ac - ab - bx + bx.$$

And since $ab-ab=o$, in the first member, and $-bx+bx=o$ in the second, the equation is reduced to

$$ax+bx=ac-ab,$$

in which it will be perceived that the quantities ab and bx have been transposed, with a change of their signs.

From what has been shown, it will appear, that, if the same quantity be in both members of the equation, and have the same sign in each, it may be left out of the equation; for, if in the equation $x+a=b+a$, we transpose a to the second member, we shall have,

$$x=b+a-a, \text{ or } x=b.$$

Rule II.

(31.) *If one or more of the terms of an equation are fractions; the equation may be cleared of fractions, by multiplying both members by the denominators of the fractions in succession, or by the least common multiple of all the denominators.*

Take the equation

$$\frac{3x}{5}+\frac{5}{2}+11=\frac{2x}{8}+17.$$

Multiplying both members by 5, we have,

$$3x+\frac{25}{2}+55=\frac{10x}{8}+85;$$

multiplying by 2,

$$6x+25+110=\frac{20x}{8}+170;$$

multiplying by 8,

$$48x+200+880=20x+1360.$$

In this operation, it is evident that the equation has not been destroyed, since, in each multiplication, both members were multiplied by the same number, (Ax. 3d.)

The equation might have been more easily cleared of fractions by the second method mentioned in the rule. The least common multiple of the denominators 5, 2, and 8, is 40; and if we multiply both members of the equation,

$$\frac{3x}{5}+\frac{5}{2}+11=\frac{2x}{8}+17,$$

by 40, we shall have the equation,

$$24x+100+440=10x+680.$$

Note.—To multiply by the common multiple, divide the common multiple by the denominator of each fraction, and multiply the quotient by the numerator; and multiply the entire terms by the whole multiple.

Examples.

1. Let the equation $\frac{2ax}{a^2-b^2}-x=\frac{2b}{a+b}+b$ be cleared of fractions.

The least common multiple of the denominators in this equation is a^2-b^2. Multiplying both members of the equation by this quantity, we have,

$$2ax-x(a^2-b^2)=2b(a-b)+b(a^2-b^2).$$

In this operation, the denominator of the first term of the first member is removed, for, $\frac{2ax}{a^2-b^2}\times a^2-b^2=2ax$ (23). The first term of the second member having $a+b$ for its denominator, we divide a^2-b^2 by $a+b$, and obtain $a-b$ for a quotient, which is multiplied into the numerator, and gives for a product, $2b(a-b)$; the other terms being entire quantities, are multiplied by the whole multiple.

Performing the operations indicated, the equation becomes

$$2ax-a^2x+b^2x=2ab-2b^2+a^2b-b^3.$$

2. Let the equation $\frac{x-2}{3a^2}-\frac{x+2}{2a^2-1}=3ab+\frac{c}{2a}$, be cleared of fractions.

The least common multiple of the denominators is $12a^4-6a^2$; dividing this by each denominator, and indicating the multiplication of the quotient by the numerators respectively, we have,

$$(x-2)(4a^2-2)-(x+2)6a^2=3ab(12a^4-6a^2)+c(6a^3-3a);$$

and performing the operations indicated,

$$4a^2x-8a^2-2x+4-6a^2x-12a^2=36a^5b-18a^3b-6a^3c+3ac$$

3. Clear of fractions the equation

$$\frac{2x}{3}+\frac{x}{2}+\frac{5}{6}=\frac{5x}{8}+\frac{7x}{9}-2.$$

Here the least common multiple of the denominators is 72; and multiplying both members by 72, we have

$$48x+36x+60=45x+56x-144.$$

Rule III.

(32.) *When the equation is reduced to such a form, that one member is composed of terms containing the unknown quantity; and the other, of terms containing known quantities only; the value of the unknown quantity may be found by dividing both members by the coefficient of the unknown quantity, if the equation be numerical, or, by the algebraic sum of the quantities into which the unknown quantity is multiplied, if the quotient be literal* (Ax. 4th.)

Examples.

1. Find the value of x from the equation,

$$12x=196.$$

Dividing both members by 12, we have,

$$x=\frac{196}{12}=16\tfrac{1}{3}.$$

2. Find the value of x from the equation,

$$3ax=15a^2.$$

Dividing by $3a$, we obtain,

$$x=\frac{15a^2}{3a}=5a.$$

3. Find the value of x from the equation,

$$ax+bx=2c.$$

The first member of this equation is the same as $x(a+b)$; dividing therefore by $a+b$, we have,

$$x=\frac{2c}{a+b}.$$

4. Find the value of x from the equation,

$$3ax+bx-3bcx=4b^3-2bc.$$

This equation may be put under the form

$$x(3a+b-3bc)=4b^3-2bc;$$

and dividing by $3a+b-3bc$, we have,

$$x=\frac{4b^3-2bc}{3a+b-3bc}.$$

General Rule.

1*st*. (33.) *Clear the equation of fractions.*

2*d*. *Transpose the terms of the equation so as to bring all the terms containing the unknown quantity into one member, and all the known terms into the other.*

3*d*. *If the equation be numerical, collect into one term all the terms containing the unknown quantity, and divide both members of the equation by the coefficient of this term. If the equation be literal, form two factors of all the terms containing the unknown quantity, one of which shall be the unknown quantity itself, and divide both members of the equation by the other factor.*

Examples.

1. Given $\frac{3x}{4}-\frac{x-1}{2}=6x-\frac{20x+13}{4}$ to find x.

Multiplying by 4 to clear the equation of fractions, the equation becomes

$$3x-2x+2=24x-20x-13.$$

Bringing all the unknown terms into the first member, and the known into the second, we have,

$$3x-2x-24x+20x=-13-2;$$

and collecting the terms,

$$-3x=-15.$$

Dividing by 3, $-x=-5$, or $x=5$.

Note 1.—In clearing this equation of fractions, the terms $\frac{x-1}{2}$ and $\frac{20x+13}{4}$ have their signs changed, since the sign $-$ before the dividing line of the fraction shows that the whole fraction is to be subtracted.

Note 2.—We have here for a result $-x=-5$, which may be changed into $x=5$, since by transposing these terms we should have $5 = x$. In general, all the signs of an equation may be changed into the opposite, since the effect would be the same as transposing the entire members; thus,

$$a^2 - 2x = 3b - 2c,$$

by transposing becomes

$$2c - 3b = 2x - a^2;$$

and since the members of this last equation are equal, in whatever order they are written, we have,

$$2x - a^2 = 2c - 3b,$$

which is simply the first equation, with the signs changed.

2. Given $\frac{3ax}{a^2 - b^2} + \frac{x}{3ab} = \frac{3b^2}{a+b} + b$, to find the value of x.

Multiplying by $3a^3b - 3ab^3$, the common multiple of the denominators, to clear of fractions, we have,

$$3ax(3ab) + (a^2 - b^2)x = 3b^2(3a^2b - 3ab^2) + b(3a^3b - 3ab^3),$$

and performing the operations indicated,

$$9a^2bx + a^2x - b^2x = 9a^2b^3 - 9ab^4 + 3a^3b^2 - 3ab^4.$$

Forming two factors of the first number, one of which is x, and reducing similar terms in the second, we have,

$$x(9a^2b + a^2 - b^2) = 9a^2b^3 - 12ab^4 + 3a^3b^2;$$

and dividing by the quantity multiplied into x, we find

$$x = \frac{9a^2b^3 - 12ab^4 - 3a^3b^2}{9a^2b + a^2 - b^2}.$$

3. Given $\frac{2x+b}{2} + 2b = \frac{b(2a - x)}{a - b} + x$ to find x.

Clearing of fractions we have,

$$(2x + b)(a - b) + 2b(2a - 2b) = 2b(2a - x) + x(2a - 2b),$$

and performing the multiplication indicated, the equation becomes,

$$\overline{2ax} + ab - \overline{2bx} - b^2 + \overline{4ab} - 4b^2 = \overline{4ab} - 2bx + \overline{2ax} - \overline{2bx}.$$

In this form of the equation, the terms marked by a line drawn over them are the same in each member, and have the same sign; they may therefore be expunged, and the equation becomes, by reducing,

$$ab - 5b^2 = -2bx.$$

Transposing the entire members, we have,

$$2bx = 5b^2 - ab.$$

Dividing by $2b$,

$$x = \frac{5b^2 - ab}{2b} = \frac{5b - a}{2}.$$

4. Given $\frac{x - 1}{7} + \frac{23 - x}{5} = 7 - \frac{4 + x}{4}$ to find x.

Multiplying both members by $5 \times 7 \times 4 = 140$, we have,

$$20x - 20 + 644 - 28x = 980 - 140 - 35x;$$

transposing,

$$20x - 28x + 35x = 980 - 140 + 20 - 644,$$

and reducing, $27x = 216.$

Dividing by 27, $x = \frac{216}{27} = 8.$

5. Given $x - \frac{3x - 3}{5} + 4 = \frac{20 - x}{2} - \frac{6x - 8}{7} + \frac{4x - 4}{5}$ to find x.

Multiplying both members of the equation by 70, the least common multiple of the denominators, we have,

$$70x - 42x + 42 + 280 = 700 - 35x - 60x + 80 + 56x - 56.$$

Transposing and reducing,

$$67x = 402,$$

and, $x = \frac{402}{67} = 6.$

NOTE 3.—In the resolution of equations, it is often convenient to depart from the rule given, and take advantage of every circumstance that presents itself for simplifying the quantities which enter into the equation, and also the equation itself; observing always to perform the same operation upon both members, that the equality may not be destroyed.

6. Given $\frac{a(c^2 + x^2)}{cx} = ab + \frac{ax}{c}$ to find the value of x.

Since a is a factor in each of the terms of this equation, it can be simplified by dividing both members by a, and the equation becomes,

$$\frac{c^2 + x^2}{cx} = b + \frac{x}{c}.$$

Clearing of fractions we have,

$$c^2 + \overline{x^2} = bcx + \overline{x^2}.$$

Reducing $c^2 = bcx;$

whence, $bcx = c^2$;

and $x = \frac{c^2}{bc} = \frac{c}{b}$.

7. Given $\frac{x^2 + b}{bx - ab} = \frac{a^2 + b}{bx - ab} + 1$.

Clearing of fractions we have,

$$x^2 + b = a^2 + b + bx - ab.$$

Expunging b, which is common to both members of the equation, and transposing a^2 to the first member, we have,

$$x^2 - a^2 = bx - ab, \text{ or } (x - a)b.$$

Dividing both members of this equation by $x - a$, we obtain,

$$x + a = b,$$

and, $x = b - a$.

8. Given $\frac{x - ax}{x - 1} = \frac{x^2 - 2a}{a} + 2$.

Clearing of fractions we have,

$$ax^2 - a^2x = x^2 - 2ax - x + 2a + 2ax - 2a;$$

and reducing, $ax^2 - a^2x = x^2 - x$.

Dividing both members by x, the equation becomes,

$$ax - a^2 = x - 1,$$

and, $ax - x = a^2 - 1$,

or, $x(a - 1) = a^2 - 1$.

Whence, $x = \frac{a^2 - 1}{a - 1}$, or $a + 1$.

Examples for Practice.

9. Given $x + \frac{x}{2} + \frac{x}{3} = 11$ to find x. Ans. $x = 6$.

10. Given $\frac{x}{5} + \frac{x}{4} + \frac{x}{3} = \frac{x}{2} + 17$ to find x. $x = 60$.

11. Given $4x - 20 = \frac{3x}{7} + \frac{110}{7}$ to find x. $x = 10$.

12. Given $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} = \frac{1}{2}$ to find x. $x = \frac{6}{7}$.

13. Given $3x + \frac{1}{9} = \frac{x + 3}{3}$ to find x. $x = \frac{1}{3}$.

14. Given $2x - \frac{x+3}{3} + 15 = \frac{12x+26}{5}$ to find x.

Ans. $x = 12$.

15. Given $\frac{x-2}{2} + \frac{x}{3} = 20 - \frac{x-6}{2}$ to find x.

$x = 18$.

16. Given $2ax + b = 3cx + 4a$ to find x.

$x = \frac{4a-b}{2a-3c}$.

17. Given $5ax + 2b + 4bx = 2x + 5c$ to find x.

$x = \frac{5c-2b}{5a+4b-2}$.

18. Given $\frac{3x-3}{4} - \frac{3x-4}{3} = 5\frac{1}{3} - \frac{27+4x}{9}$ to find x.

$x = 9$.

19. Given $\frac{3a+x}{x} - 5 = \frac{6}{x}$ to find x. $x = \frac{3a-6}{4}$.

20. Given $\frac{5ab}{6} + \frac{4ac}{5} - \frac{2cx}{3} = \frac{3ac}{4} + 2ab - 6cx$ to find x.

$x = \frac{70ab - 3ac}{320c}$.

21. Given $\frac{a^2x}{b-c} - dc = bx - ac$ to find x.

$x = \frac{c(b-c)(d-a)}{a^2 - b^2 + bc}$.

22. Given $\frac{cx^m}{a+bx} = \frac{dx^m}{e+fx}$ to find x. $x = \frac{ad-ce}{cf-bd}$.

23. Given $(a+x)(b+x) - a(b+c) = \frac{a^2c}{b} + x^2$ to find x.

$x = \frac{ac}{b}$.

24. Given $\frac{9x+20}{36} = \frac{4x-12}{5x-4} + \frac{x}{4}$ to find x. $x = 8$.

25. Given $\frac{9x-7}{2} + \frac{11}{x} + \frac{4x+12}{2} = \frac{75x}{6} - \frac{7x+9-3}{2x}$
$+ \frac{14}{x}$ to find x. $x = 1$.

26. Given $\frac{ax+2x}{5a} = \frac{a^2+4a+4}{4b}$ to find x. $x = \frac{5a^2+10a}{4b}$.

27. Given $9ax^3 - 15abx^2 = 6ax^3 + 12ax^2$ to find x.

$$x = 5b + 4.$$

28. Given $\frac{a-b}{x} + \frac{2a^2}{ax+bx} = \frac{9a^4-4b^4}{a+b} + \frac{b^2}{ax+bx}$ to find x.

$$x = \frac{1}{3a^2 + 2b^2}.$$

29. Given $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{g}{x}$ to find x.

$$x = \frac{bdfg}{adf + cbf + bde}.$$

30. Given $\frac{7x-43}{12} + 13\frac{1}{2} - \frac{5+4x}{6} = 256 - \frac{3x-12}{9} - \frac{5x+29}{8} - 12x$ to find x. $x = 19.$

SOLUTION OF PROBLEMS PRODUCING SIMPLE EQUATIONS.

(34.) In every problem there is something given and known, and something unknown to be determined; and it is by means of the relations of the known quantities to the unknown, that we are enabled to arrive at the solution.

The solution of a problem, by means of algebra, consists of two parts; the first is to express in an equation the conditions of the problem; and the second, to reduce that equation, so as to find the value of the unknown quantity.

For expressing, in an equation, the conditions of a problem, the following rule may be observed.

RULE.

Express, by means of a letter, the unknown quantity, and perform upon it all the arithmetical operations which would be necessary to verify the result, were the problem already solved. This will form an equation, which, reduced according to the foregoing rules, will give the value of the unknown quantity.

EXAMPLES.

1. To find a number, such that the sum of its third and fourth parts, shall be equal to its half, plus 18.

If we suppose the number already found, and we wished to verify it, we should add its third and fourth parts together, and see if the sum was equal to its half, increased by 13.

Putting x for the number, we proceed in the same manner: $\frac{x}{3}$ is its third part, $\frac{x}{4}$ its fourth part, and $\frac{x}{2}$ its half. We should then have the following equation:—

$$\frac{x}{3}+\frac{x}{4}=\frac{x}{2}+13.$$

Clearing of fractions, $4x+3x=6x+156$.
Transposing, $7x-6x=156$;
and reducing, $x=156$.

Verification.

$$\frac{156}{3}+\frac{156}{4}=\frac{156}{2}+13.$$

or, $52+39=78+13$.
Whence, $91=91$, which shows the result to be correct.

2. The sum of the two numbers is 30, the difference of their squares is 120; required the numbers.

Let x equal the least of the two numbers; then since their sum is 30, the greater must be $30-x$. The square of $30-x$ is $900-60x+x^2$. Subtracting from this, the square of x, the remainder, by the conditions of the problem, must be 120. We have, therefore, the following equation:—

$$900-60x+x^2-x^2=120,$$

or, $900-60x=120$.
Dividing by 60, $15-x=2$,
or, $-x=-13$.
The least number is 13, and the greater $30-13=17$.

Verification.

$$(17)^2-(13)^2=120,$$

or, $289-169=120$,
or, $120=120$.

It is sometimes convenient to substitute letters in the place of the known quantities.

In the preceding problem, let $30 = a$, and $120 = b$; then we have for an equation,

$$(a - x)^2 - x^2 = b;$$

and by squaring $a - x$,

$$a^2 - 2ax + x^2 - x^2 = b,$$

or, $$a^2 - 2ax = b.$$

Transposing, $$-2ax = b - a^2.$$

Changing the signs, $$2ax = a^2 - b;$$

whence, $$x = \frac{a^2 - b}{2a} = \frac{a}{2} - \frac{b}{2a}.$$

Or if we substitute the values of a and b;

$$x = \frac{30}{2} - \frac{120}{60} = 15 - 2 = 13.$$

Let x be the greater number, and we shall have the equation, $$x^2 - (a - x)^2 = b,$$

or, $$x^2 - a^2 + 2ax - x^2 = b,$$

and, $$-a^2 + 2ax = b,$$

$$2ax = b + a^2,$$

$$x = \frac{a^2 + b}{2a} = \frac{a}{2} + \frac{b}{2a};$$

and restoring the values of a and b,

$$x = \frac{30}{2} + \frac{120}{60} = 15 + 2 = 17.$$

3. A gentleman meeting four poor persons, distributed five shillings among them in the following manner:—to the second he gave twice, to the third, three times, and to the fourth, four times, as much as to the first. What did he give to each?

Let $x =$ the pence he gave to the first;
then $2x =$ the pence he gave to the second,
$3x =$ the pence he gave to the third,
and $4x =$ the pence he gave to the fourth.

But the sum of the whole is five shillings, or sixty pence;

therefore, $x + 2x + 3x + 4x = 60$;
reducing, $10x = 60$,
and $x = 6$.

Wherefore he gave 6, 12, 18, and 24 pence to them respectively.

4. It is required to divide the number 99 into five such parts, that the first may exceed the second by 3, be less than the third by 10, greater than the fourth by 9, and less than the fifth by 16.

Let x = the first part;
then $x-3$ = the second,
$x+10$ = the third,
$x-9$ = the fourth,
$x+16$ = the fifth.

The sum of all the parts being 99, we have,

$$x-3+x+10+x-9+x+16=99.$$

Reducing $5x+14=99$;
transposing, $5x=99-14=85$;
whence, $x=17$.

The parts are, therefore, 17, 14, 27, 8 and 33.

5. A courier, who travels 60 miles a day, had been despatched 5 days, when a second was sent to overtake him, in order to which he must travel 75 miles a day; in what time will he overtake him?

Let x = the number of days the second courier travels; then, since the first had been despatched 5 days before, he must travel $x+5$ days. But they travel the same distance; hence, since the first travels 60 miles per day, and the second 75, we shall have the equation,

$$60(x+5)=75x,$$

or, $$60x+300=75x.$$

Transposing, and changing the signs, we have,

$$15x=300,$$

and, $$x=20.$$

6. A merchant sells two pieces of cloth, the lengths of which are as 3 to 4. The shorter piece he sells for 13 cents more per yard than the longer, and the sum he receives for it, is to the sum he receives for the longer, as 7 to 8. At what price per yard were they sold?

Let x = the price per yard of the shorter;
then, $x-13$ is the price of the other.

Then, since their lengths are as 3 to 4, we shall have, the sum received for the first, to the sum received for the second, as, $3\times x$, is to $4\times(x-13)$.

But these sums are in the ratio of 7 to 8; therefore,

$$3x : 4(x-13) = 7 : 8.$$

And by multiplying means and extremes, we have (Arith. 65),

$$24x = 28(x-13) = 28x - 364.$$

Transposing and changing the signs,

$$4x = 364,$$

$$x = 91.$$

91, and $91 - 13 = 78$, are the prices respectively.

7. A merchant commences business with a certain capital, and borrows as much more. At the end of each year he finds that his expenses have exceeded his income by $1000, and borrows as much as he has remaining. Four years having passed in this way, his whole capital is exhausted. What capital had he when he began?

Let x = the capital with which he commenced.

Then, since he borrows as much as he has, and loses $1000 the first year, he will have at the end of the year $2x - 1000$.

In like manner, doubling this sum and subtracting $1000, he will have, at the end of the second year, $4x-2000-1000$; at the end of the third, $8x-4000-2000-1000$; and at the end of the fourth, $16x-8000-4000-2000-1000$; but his capital being now exhausted, this last sum is equal to nothing; we have, therefore,

$$16x-8000-4000-2000-1000 = 0.$$

Transposing and reducing, $16x = 15000$,

and, $x = \frac{15000}{16} = 937{,}5.$

8. Two workmen, A and B, are employed together for 50 days, at 60 cents per day each. A spent 6 cents less per day than B, and at the end of 50 days, he found that he had saved twice as much as B, and the expense of two days over. What did each spend per day?

Let x = what A spent per day; then $60 - x$ is what he saves per day; and since B spends 6 cents more, he saves but $60 - x - 6$, or $54 - x$. By the conditions of the question, what A saves in 50 days, *minus* the expense of two days, is equal to twice what B saves in 50 days; therefore,

$$50(60-x) - 2x = 2 \times 50(54-x),$$

or, $3000 - 50x - 2x = 5400 - 100x,$

and reducing, $48x = 2400,$

$$x = \frac{2400}{48} = 50.$$

A therefore spent 50 cents per day, and B 56.

9. A bookseller sells two books, one containing 100 sheets for 10 shillings, and the other, containing 50 sheets for 6 shillings, each being bound at the same price. What was that price?

Let x = the price;

then, $10 - x : 6 - x = 100 : 50.$

Multiplying the means and extremes of this proportion, we have (Arith. 65),

$$500 - 50x = 600 - 100x:$$

by transposition, $50x = 100,$

$x = 2s$, the price of binding.

10. A man wishing to enclose a piece of ground with palisadoes, finds, that if he sets them a foot asunder, he will have too few by 150, but if he sets them a yard asunder, he will have too many by 70. How many has he?

Let x = the nunber;

then, $x - 70 : x + 150 = 1 : 3.$

Comparing the difference of consequents and antecedents, with the antecedents (Arith. 69), we have,

$$x - 70 : 220 = 1 : 2.$$

Dividing consequents by 2 (Arith. 68),

$$x - 70 : 110 = 1 : 1,$$

therefore, $x - 70 = 110,$

and, $x = 180.$

11. A market woman bought a certain number of eggs at two for a penny, and as many at three for a penny, and sold them out at five for two pence, and lost four pence by so doing. How many did she buy and sell?

Let x = the number of eggs of each price;

then, $\frac{x}{2}$ = the cost of the first.

and, $\frac{x}{3}$ = the cost of the second.

But, since she sold five for two pence, $\frac{2}{5}$ is the price of each, which, multiplied by $2x$, the whole number of eggs, gives for the amount $\frac{2}{5} \times 2x$, or $\frac{4x}{5}$, which is, by the conditions of the question, four pence less than the cost; we have, therefore

$$\frac{x}{2} + \frac{x}{3} - \frac{4x}{5} = 4;$$

whence, $15x + 10x - 24x = 120$,

and, $x = 120$; and hence the whole number is 240.

12. There is a certain number, consisting of 2 digits, the sum of these digits is 6, and if 18 be added to the number itself, the digits will be inverted; what is the number?

Let x = the left hand digit.

Then, $6 - x$ = the other; and since each figure in a number is ten times greater when removed one place to the left, the number itself will be

$10x + 6 - x$, and $10(6 - x) + x$ is the number with its digits inverted; we shall then have,

$$10x + 6 - x + 18 = 10(6 - x) + x;$$

reducing, $18x = 36$;

whence, $x = 2$,

and $6 - x$, the other digit $= 4$.

Hence the number is 24, and this result is verified by adding 18 to 24, which gives 42, the digits inverted.

13. A merchant drew every year upon the stock he had in trade, the sum of a dollars for the expense of his family. His profits each year were the nth part of what remained after this reduction; but at the end of three years, he finds that his whole stock is exhausted; how much had he at the beginning?

Let x = the stock in trade;

then $x - a$ = what remains after a dollars is deducted;

and $\frac{x - a}{n}$ is the profits of the first year; adding this to $x - a$, the sum will be the stock at the end of the first year;

wherefore, $x - a + \frac{x - a}{n} = \frac{nx - na + x - a}{n}$, the stock is at the end of the first year.

Subtracting from this, a, for the expenses of the second year, we have,

$$\frac{nx-na+x-a}{n}-a=\frac{nx-na+x-a-na}{n}=\frac{nx-2na+x-a}{n}.$$

The profits of the second year is the nth part of this sum, or

$$\frac{nx-2na+x-a}{n^2}.$$

Adding this amount to the stock at the beginning of the year, we have,

$$\frac{nx-2na+x-a}{n}+\frac{nx-2na+x-a}{n^2},$$

$$=\frac{n^2x+2nx+x-2n^2a-3na-a}{n^2},$$

for the amount at the close of the second year.

Subtracting a from this, we have,

$$\frac{n^2x+2nx+x-2n^2a-3na-a}{n^2}-a$$

$$=\frac{n^2x+2nx+x-3n^2a-3na-a}{n^2}.$$

Adding to this remainder its nth part, we have for the stock at the end of the third year,

$$\left\{\begin{array}{l}\dfrac{n^2x+2nx+x-3n^2a-3na-a}{n^2}\\[2ex] +\dfrac{n^2x+2nx+x-3n^2a-3na-a}{n^3};\end{array}\right.$$

which becomes, by bringing the fractions to a common denominator, and reducing similar terms,

$$\frac{n^3x+3n^2x+3nx+x-3n^3a-6n^2a-4na-a}{n^3}.$$

But, by the conditions of the problem, the whole stock is exhausted at the end of the third year; therefore,

$$\frac{n^3x+3n^2x+3nx+x-3n^3a-6n^2a-4na-a}{n^3}=0.$$

Clearing of fractions and transposing,

$$n^3x + 3n^2x + 3nx + x = 3n^3a + 6n^2a + 4na + a$$

$$x(n^3 + 3n^2 + 3n + 1) = 3n^3a + 6n^2a + 4na + a$$

$$x = \frac{3n^3a + 6n^2a + 4na + a}{n^3 + 3n^2 + 3n + 1}$$

This result may be simplified, by observing that both the numerator and denominator of the fraction may be divided by $n + 1$, and we have,

$$x = \frac{a(3n^2 + 3n + 1)(n + 1)}{(n^2 + 2n + 1)(n + 1)} = \frac{a(3n^2 + 3n + 1)}{n^2 + 2n + 1}.$$

This same result might have been obtained by a shorter method; for, if at the end of the third year nothing remained, the whole stock must have been exhausted in deducting from the stock at the beginning the sum of a dollars, for the expenses of this year; for had any thing remained after this sum was deducted, what remained, *plus* the nth part of it, must have remained at the close of the year, and the stock would not have been exhausted; whence we have,

$$\frac{n^2x + 2nx + x - 3n^2a - 3na - a}{n^2} = 0,$$

and $$n^2x + 2nx + x = 3n^2a + 3na + a,$$

and $$x = \frac{3n^2a + 3na + a}{n^2 + 2n + 1}, \text{ or } \frac{a(3n^2 + 3n + 1)}{n^2 + 2n + 1}.$$

14. There are two numbers whose sum is the sixth part of their product, and the greater is to the less as 3 to 2. Required those numbers. Ans. 15 and 10.

15. Find a number, such, that whether it be divided into two or three equal parts, the continued product of the parts shall be equal to the same quantity. Ans. $6\frac{3}{4}$.

16. There are two numbers whose sum is 37, and if three times the less be subtracted from four times the greater, and this difference divided by 6, the quotient will be 6. What are the numbers? Ans. 21 and 16.

17. What number is that to which, if 20 be added, and from $\frac{2}{3}$ of the sum 12 be subtracted, the remainder will be 10? Ans. 13.

18. Divide the number 72 into three parts, so that one-half the first shall be equal to the second, and three-fifths of the second equal to the third. Ans. 40, 20, and 12.

19. A person after spending one-fifth of his income, plus \$10, had then remaining one-half of it, plus \$35. Required his income. Ans. \$150.

20. A and B engaged in trade, A with \$240, and B with \$96; A lost twice as much as B; and upon settling their accounts it appeared, that A had three times as much remaining as B. How much did each lose? Ans. A lost \$96, B \$48.

21. A person being asked the hour, answered that it was between five and six, and the hour and minute hand were together. What was the time?
Ans. 5h. 27m. $16\frac{4}{11}$ sec.

22. It is required to divide the number 34 into two such parts, that the difference between the greater and 18, shall be to the difference between the less and 18, as 2 to 3.
Ans. 22 and 12.

23. A cistern into which water was let by two cocks, A and B, will be filled by both, running together, in 12 hours, and by the cock A alone, in 20 hours. In what time will it be filled by the cock B alone? Ans. 30 hours.

24. Two persons, A and B, have the same annual income, A saves one-fifth of his; but B, by spending \$80 per annum more than A, at the end of 4 years finds himself \$220 in debt. What was their annual income? Ans. \$125.

35. A shepherd, in time of war, was plundered by a party of soldiers, who took one-fourth of his flock and one-fourth of a sheep more; another took one-third of what he had left and one-third of a sheep; then a third party took one-half of what then remained and one-half a sheep. After which he had but twenty-five sheep left. How many had he at first?
Ans. 103.

26. A person passed $\frac{1}{6}$th of his age in childhood, $\frac{1}{12}$th in youth, $\frac{1}{7}$th + 5 in matrimony; he *then* had a son whom he survived four years, and who reached only one-half the age of his father. At what age did this person die? Ans. 84.

27. A trader maintained himself for three years, at the expense of £50 a year, and in each of those years augmented that part of his stock which was not so expended by one-third thereof. At the end of the third year his original stock was doubled. What was that stock? Ans. £740.

28. Two men, at the distance of 150 miles, set out to meet each other; one goes three miles in the time that the

other goes seven. What part of the distance does each travel? Ans. one 45, and the other 105.

29. Out of a certain sum a man paid his creditors $96, half of the remainder he lent a friend; he then spent one-fifth of what now remained, and after all these deductions had one-tenth of his money left. How much had he at first? Ans. $128.

30. The expense of paving a square court at 50 cents per square yard, is the same as that of surrounding it with an iron fence at $1 75 per foot. How many square feet does it contain? Ans. 15,876.

SOLUTION OF SIMPLE EQUATIONS WHICH INVOLVE MORE THAN ONE UNKNOWN QUANTITY.

(35.) When, by the conditions of a problem, two or more unknown quantities are to be determined, it is necessary that there should be as many independent equations as there are unknown quantities, otherwise the problem would be indeterminate.

To illustrate this take the following example:—

A laborer having worked for a person twelve days, and having with him, during the first seven days, his son, received 100 shillings; he afterwards worked eight days more, five of which he had his son with him, and received 68 shillings. How much did he earn per day himself, and how much did his son earn?

Let x = the daily wages of the man, and y = those of his son; then $12x$ is the whole of the father's wages, and $7y$ the son's; and the sum of these is, by the conditions of the problem, 100 shillings; we have, therefore, the equation

$$12x+7y=100.$$

But this equation cannot be determined, since the value of x depends on that of y. If, for instance, $y=4$, then

$$12x+7\times4=100,$$
$$12x+28=100,$$
$$12x=72,$$
$$x=6;$$

but if $y=6$, then we shall have $12x+42=100$, and $x=\frac{58}{12}=4\frac{5}{6}$. But if we have another condition and another equation, we can determine the value of x by combining the two equations together, and there will be but one value of either of the unknown quantities that will answer the conditions of the problem. This other equation can be derived from the problem; for the laborer worked eight days afterwards at the same price, and had with him his son five of these days, and received 68 shillings. This condition gives the equation

$$8x + 5y = 68.$$

By the first condition we had

$$12x + 7y = 100.$$

Now, in order to find the value of x and y, let it be observed, that an equation is not destroyed by multiplying both terms by the same quantity. Multiplying, therefore, the first of these equations by 7, and the second by 5, they become,

$$7 \times (8x + 5y = 68) = 56x + 35y = 476$$
$$5 \times (12x + 7y = 100) = 60x + 35y = 500.$$

It is evident, moreover, that if one equation be subtracted from another, term by term, the result will be an equation, since it is only subtracting equals from equals, and the remainders must therefore be equal.

If, now, we subtract the first of these last equations from the second, we shall have,

$$\begin{array}{l} 60x + 35y = 500 \\ 56x + 35y = 476 \\ \hline 4x \qquad\quad = 24, \end{array}$$

an equation, which has but one unknown quantity, and which gives $x = 6.$

Substituting this value of x, in the equation $12x + 7y = 100$, we have,

$$12 \times 6 + 7y = 100, \text{ or } 72 + 7y = 100,$$
$$\text{and,} \qquad 7y = 100 - 72 = 28$$
$$\text{whence,} \qquad y = 4.$$

These values of x and y, introduced into the original equations, are the only numbers that will satisfy the conditions of the problem. By substituting 6 in the place of x, and 4 in the place of y, we have,

$$12 \times 6 + 7 \times 4 = 100, \text{ or } 72 + 28 = 100,$$
$$\text{and, } 8 \times 6 + 5 \times 4 = 68, \text{ or } 48 + 20 = 68.$$

Now, if in the first equation, we should make $x = 5$, and $y = 5\frac{5}{7}$, we should have, $12 \times 5 + 7 \times 5\frac{5}{7} = 100$, and first equation would be verified; but these same values will not verify the other equation, since they would give,

$$8\times5+5\times5\tfrac{5}{7}=68\tfrac{4}{7}.$$

By an extension of this same reasoning, it might be shown, that where there are three unknown quantities, there must be three equations; and in general, that the number of unknown quantities and the number of equations must be equal.

The process of casting out an unknown quantity as in the preceding example, is called *elimination.*

Rules for the resolution of equations containing two or more unknown quantities.

Rule I.

(36.) *If there be two equations and two unknown quantities, multiply or divide the given equations by such quantities as will make the coefficients of the same unknown quantity equal in both equations; then, if the signs of this quantity be alike, subtract one equation from the other; and if the signs be unlike, add the equations together; the resulting equation will contain but one unknown quantity, the value of which may be found by the preceding rules.*

Substitute this value in one of the original equations, which will then contain but one unknown quantity; and the resolution of which will give the value of that unknown quantity.

If there be more than two equations, and more than two unknown quantities, eliminate one of the unknown quantities by comparing one of the equations with all the rest, according to the preceding rule: there will result a number of equations one less, containing one unknown quantity less than the given equations.

Proceed in the same manner with this new set of equations, and continue the operation till an equation is obtained which contains but one unknown quantity.

Having found the value of this, substitute it in one of the equations which contains but two unknown quantities, and the value of a second unknown quantity can be determined; then substitute these two values in an equation containing three unknown quantities; proceed in this way till the values of all the unknown quantities are determined.

RULE II.

Observe which of the unknown quantities is least involved, and let the value of that quantity be found in each equation, by considering the rest as known.

Let the values thus found be put equal to each other, and new equations will be obtained, out of which that quantity will be excluded. Repeat the operation with these new equations till an equation is obtained, which contains but one unknown quantity.

RULE III.

Find the value of one of the unknown qnantities in one equation in terms of the others, and substitute this value in each of the other equations; and with the new equations thus arising, repeat the operation till you arrive at an equation with only one unknown quantity.

EXAMPLES.

1. Given $\left\{\begin{array}{l} 5x+2y=19 \\ 7x-6y=9 \end{array}\right\}$ to find x and y.

If we multiply the first of these equations by 3, the coefficients of y will be equal, and the equations become,

$$15x+6y=57,$$

and, $$7x-6y=9.$$

To eliminate y, we add these equations (the signs of the terms containing y being unlike), and there results the equation, $$22x=66,$$

from which we find, $x=3$.

Substituting this value of x, in the equation $5x+2y=19$, it becomes,

$$15+2y=19;$$

whence, $$2y=19-15=4,$$

and, $$y=2.$$

By Rule II.

From the equation $5x+2y=19$, we find

$$y=\frac{19-5x}{2};$$

and from the equation $7x - 6y = 9$, we have,

$$y = \frac{7x-9}{6}.$$

Putting these values of y equal to each other we have the equation

$$\frac{7x-9}{6} = \frac{19-5x}{2}.$$

Clearing of fractions, $14x - 18 = 114 - 30x$.
Transposing and reducing, $44x = 132$,
and, $x = 3$.

And since $y = \frac{7x-9}{6}$, we have, by putting for x, its value

$$y = \frac{21-9}{6} = \frac{12}{6} = 2.$$

By Rule III.

From the equation $5x + 2y = 18$, we have,

$$y = \frac{19-5x}{2};$$

and substituting this value of y, in the equation $7x - 6y = 9$, we have,

$$7x - 6\left(\frac{19-5x}{2}\right) = 9;$$

and performing the operations indicated,

$$7x - 57 + 15x = 9;$$

reducing, $22x = 66$,
and $x = 3$.

Putting this value of x in the equation $7x - 6y = 9$, we have,

$$21 - 6y = 9.$$

Transposing and reducing

$$-6y = -12;$$

whence, $y = 2$.

2. Given $\left\{\begin{array}{l} ax + by = n \\ cx + dy = m \end{array}\right\}$ to find x and y.

To eliminate x, multiply the first equation by c, and the second by a; they will then become,

$$acx + bcy = cn,$$
$$acx + ady = am.$$

Subtracting the second of these equations from the first, there remains the equation

$$bcy - ady = cn - am,$$

or, $$y(bc - ad) = cn - am;$$

whence, $$y = \frac{cn - am}{bc - ad}.$$

To find the value of x, multiply the first of the two given equations by d, and the second by b; they become,

$$adx + bdy = dn,$$
$$bcx + bdy = bm.$$

Subtracting the first of these equations from the second, there remains,

$$bcx - adx = bm - dn,$$

or, $$x(bc - ad) = bm - dn;$$

whence, $$x = \frac{bm - dn}{bc - ad}.$$

The value of x may also be obtained by substituting the value of y already found in one of the original equations; thus, taking the equation $ax + by = n$, and putting for y its value, we have,

$$ax + b\left(\frac{cn - am}{bc - ad}\right) = n.$$

Transposing, $$ax = n - b\left(\frac{cn - am}{bc - ad}\right).$$

Reducing the second member of this equation to a fraction, we have,

$$ax = \frac{bcn - adn - bcn + abm}{bc - ad} = \frac{abm - adn}{bc - ad}.$$

Dividing both members by a, we find,

By Rule II.

The first equation gives,

$$y=\frac{n-ax}{b};$$

and the second, $y=\frac{m-cx}{d}$;

we have, therefore,

$$\frac{n-ax}{b}=\frac{m-cx}{d}.$$

Clearing of fractions,

$$dn-adx=bm-bcx.$$

Transposing, $bcx-adx=bm-dn$,

or, $x(bc-ad)=bm-dn$;

whence, $x=\frac{bm-dn}{bc-ad}$.

The value of y may be found in like manner, or by substituting the value of x in one of the original equations.

By Rule III.

From the first equation we find $y=\frac{n-ax}{b}$. Substituting this value of x in the second, we have,

$$cx+d\left(\frac{n-ax}{b}\right)=m,$$

or, $cx+\frac{dn-adx}{b}=m.$

Clearing of fractions,

$$bcx+dn-adx=bm;$$

transposing, $bcx-adx=bm-dn$;

whence, $x=\frac{bm-dn}{bc-ad}$.

The value of y may be found in like manner, or by substitution in one of the given equations.

3. Given $\left\{\begin{array}{rl}3x-2y+z&=9\\ x+7y-4z&=11\\ 8x-y-3z&=6\end{array}\right\}$ to find x, y and z.

By Rule I.

In order to eliminate z, we multiply the first equation by 4; the coefficients of z will then be equal in the first and second. The first equation, multiplied by 4, becomes,

$$12x - 8y + 4z = 36.$$

If now we add to this the equation

$$x + 7y - 4z = 11.$$

z will not be found in the result, and we shall have the equation

$$13x - y = 47.$$

To eliminate z from the third equation, we compare it with the first which we multiply by 3, and it becomes

$$9x - 6y + 3z = 27.$$

Adding to this, the third equation,

$$8x - y - 3z = 6,$$

we obtain $17x - 7y = 33.$

We have now reduced the three given equations to two, having two unknown quantities each, namely,

$$13x - y = 47,$$

and, $17x - 7y = 33.$

To eliminate y from these equations, we multiply the first by 7, and we have,

$$91x - 7y = 329.$$

Subtracting from this the second, there remains,

$$74x = 296;$$

whence, $x = 4.$

To find y, we substitute this value of x in the equation $13x - y = 47$, and we have,

$$13 \times 4 - y = 47,$$

or, $52 - y = 47;$

whence we obtain, $y = 5.$

To find z, substitute these values of x and y in an equation containing z, as,

$$8x - y - 3z = 6,$$

and it becomes $32 - 5 - 3z = 6.$

Transposing and reducing,

$$-3z = -21;$$

whence, $z = 7.$

By Rule II.

From the first equation, by transposing and reducing, we find,

$$x = \frac{9 + 2y - z}{3}.$$

From the second, $x = 11 + 4z - 7y$;
and from the third,

$$x = \frac{6 + y + 3z}{8}.$$

From the first two values of x, we derive the equation,

$$\frac{9 + 2y - z}{3} = 11 + 4z - 7y;$$

clearing of fractions,

$$9 + 2y - z = 33 + 12z - 21y;$$

and bringing the unknown quantities into the first member,

$$23y - 13z = 24.$$

By comparing the second and third values of x, we have,

$$\frac{6 + y + 3z}{8} = 11 + 4z - 7y.$$

Clearing of fractions,

$$6 + y + 3z = 88 + 32z - 56y;$$

whence, $57y - 29z = 82.$

We have now obtained two equations with two unknown quantities, upon which we operate in the same manner as before.

The first gives, $y = \frac{24 + 13z}{23}$,

and the second, $y = \frac{82 + 29z}{57}$.

Putting these values of y equal to each other, we have the equation,

$$\frac{24 + 13z}{23} = \frac{82 + 29z}{57}.$$

Clearing of fractions,

$$1368 + 741z = 1886 + 667z;$$

whence, $74z = 518,$

and, $z = 7.$

Substituting this value of z in the equation,

$$y = \frac{24 + 13z}{23},$$

we have, $$y = \frac{24 + 91}{23} = \frac{115}{23} = 5.$$

To find x, we have,

$$x = 11 + 4z - 7y,$$

or, $$x = 11 + 28 - 35 = 4.$$

By Rule III.

From the second of the given equations, we find,

$$x = 11 + 4z - 7y.$$

Putting this value of x, in the first equation, it becomes,

$$3(11 + 4z - 7y) - 2y + z = 9;$$

or, by performing the operations indicated, and reducing

$$13z - 23y = -24.$$

Again putting the value of x in the third equation, in the place of x, we have,

$$8(11 + 4z - 7y) - y - 3z = 6;$$

and reducing, $$29z - 57y = -82;$$

or, by changing the signs of both members,

$$57y - 29z = 82.$$

From the equation $13z - 23y = -24$, we obtain

$$z = \frac{23y - 24}{13}.$$

And substituting this value of z in the preceding equation, it becomes,

$$57y - 29\left(\frac{23y - 24}{13}\right) = 82,$$

or, by performing the operations indicated,

$$57y - \frac{667y + 696}{13} = 82.$$

Clearing of fractions,

$$741y - 667y + 696 = 1066.$$

Reducing, $$74y = 370,$$

and, $$y = 5.$$

The values of x and z may be found as in the first and second rules.

Of the three rules for the resolution of equations containing two or more unknown quantities, the first is generally to be preferred, on account of its simplicity; but there are cases where the operation may be abridged by the use of the others. It often occurs that one unknown quantity is most easily eliminated by one rule, and another by another rule; and thus two, or all the rules, be used in reducing one set of equations.

4. For an example of this kind, take the equations

$$2x + y = 10. \quad [1]$$
$$x - 3y + 2u = 8. \quad [2]$$
$$5z - 2u = 5. \quad [3]$$
$$2z - y = 4. \quad [4]$$

In this example, x may be eliminated by the second rule, by putting its values in equations [1] and [2], equal to each other. The first gives

$$x = \frac{10 - y}{2};$$

and the second, $x = 8 + 3y - 2u$;

wherefore, $\frac{10 - y}{2} = 8 + 3y - 2u,$

or, $10 - y = 16 + 6y - 4u,$

and, $4u - 7y = 6. \quad [5]$

To eliminate z, we can make use of Rule III.

Its value in equation, [4] is

$$z = \frac{4 + y}{2}.$$

Substituting this in equation [3], that equation becomes,

$$5\left(\frac{4 + y}{2}\right) - 2u = 5,$$

or, $20 + 5y - 4u = 10$;

whence, $5y - 4u = -10$;

or, changing the signs,

$$4u - 5y = 10. \quad [6]$$

Now, if this last equation be subtracted from equation [5], according to rule 1st, there will remain,

$$-2y = -4,$$

or,
$$y = 2.$$

Then from equation [1], we have,

$$2x + 2 = 10,$$
$$x = 4.$$

From equation [2], $4 - 6 + 2u = 8,$
$$u = 5.$$

And from equation [4], $2z - 2 = 4,$
$$z = 3.$$

5. Given
$$\begin{cases} \frac{x}{3} - \frac{y+z}{2} + 8 = \frac{2z+3y}{4} \\ \frac{3x}{2} + \frac{3y}{2} = \frac{x+z}{2} + 12 \\ \frac{4z}{3} - \frac{x+7}{4} = \frac{2x+3y+z}{9}. \end{cases}$$

Clearing these equations of fractions, transposing and reducing, they become,

$$12z + 15y - 4x = 96,$$
$$-z + 3y + 2x = 24,$$
$$44z - 12y - 17x = 63.$$

If we multiply the second equation by 5, the coefficients of y will be equal in the first and second; the second equation by this multiplication becomes,

$$-5z + 15y + 10x = 120,$$

which, being subtracted from the first, there remains the equation, $17z - 14x = -24;$

or, by changing the signs,

$$14x - 17z = 24.$$

To eliminate y from the third equation, we multiply the second by 4, and we have,

$$-4z + 12y + 8x = 96,$$

which, added to the third, gives the equation,

$$40z - 9x = 159.$$

We have thus obtained two equations with but two unknown quantities, x and z. Multiplying the first of these by 9, and the second by 14, they become,

$$126x - 153z = 216,$$
$$560z - 126x = 2226.$$

Adding these equations together, we have,

$$407z = 2442;$$

whence, $$z = 6.$$

By substituting as in the preceding examples, we find

$$x = 9, \text{ and } y = 4.$$

6. Given $\left\{\begin{array}{l} x + y = c \\ ax - by = d \end{array}\right\}$ to find x and y.

Ans. $x = \dfrac{bc + d}{a + b}$ and $y = \dfrac{ac - d}{a + b}$.

7. Given $\left\{\begin{array}{l} 11x + 3y = 100 \\ 4x - 7y = 4. \end{array}\right.$

Ans. $x = 8$, and $y = 4$.

8. Given $\left\{\begin{array}{l} \dfrac{x}{2} + \dfrac{y}{3} = 7 \\ \dfrac{x}{3} + \dfrac{y}{2} = 8 \end{array}\right.$

Ans. $x = 6$, and $y = 12$.

9. Given $\left\{\begin{array}{l} \dfrac{x}{7} + 7y = 99 \\ \dfrac{y}{7} + 7x = 51. \end{array}\right.$

Ans. $x = 7$, and $y = 14$.

10. Given $\left\{\begin{array}{l} \dfrac{x}{2} - 12 = \dfrac{y}{4} + 8 \\ \dfrac{x + y}{5} + \dfrac{x}{3} - 8 = \dfrac{2y - x}{4} + 27. \end{array}\right.$

Ans. $x = 60.$
$y = 40.$

11. Given $\left\{\begin{array}{l} x + y = 13. \\ x + z = 14. \\ y + z = 15. \end{array}\right.$ Ans. $\left\{\begin{array}{l} z = 8. \\ y = 7. \\ x = 6. \end{array}\right.$

$$12.\ \text{Given}\ \left\{\begin{array}{l} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62. \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47. \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38. \end{array}\right. \qquad \text{Ans.}\ \left\{\begin{array}{l} x=24. \\ y=60. \\ z=120. \end{array}\right.$$

$$13.\ \text{Given}\ \left\{\begin{array}{l} x+100=y+z. \\ y+100=2x+2z. \\ z+100=3x+3y. \end{array}\right. \qquad \text{Ans.}\ \left\{\begin{array}{l} x=9\tfrac{1}{11}. \\ y=45\tfrac{5}{11}. \\ z=63\tfrac{7}{11}. \end{array}\right.$$

$$14.\ \text{Given}\ \left\{\begin{array}{r} 2x+y-2z=40. \\ 4y-x+3z=35. \\ 3u+t=13. \\ y+u+t=15. \\ 3x-y+3t-u=49. \end{array}\right. \qquad \text{Ans.}\ \left\{\begin{array}{l} x=20. \\ y=10. \\ z=\ \ 5. \\ u=\ \ 4. \\ t=\ \ 1. \end{array}\right.$$

(37.) ***Problems producing simple equations involving two or more unknown quantities.***

1. What fraction is that, to the numerator of which, if 4 be added, the value is $\frac{1}{2}$; but if 7 be added to the denominator, the value is $\frac{1}{5}$?

Let $\frac{x}{y}$ be the fraction required, then

$$\frac{x+4}{y}=\frac{1}{2}, \text{ and } \frac{x}{y+7}=\frac{1}{5}.$$

Clearing of fractions, these equations give,

$$\left.\begin{array}{l} 2x+8=y \\ 5x=y+7 \end{array}\right\} \text{ or } \left\{\begin{array}{l} 2x-y=-8 \\ 5x-y=7. \end{array}\right.$$

Subtracting the first of these from the second, we obtain

$$3x=15;$$

whence,
$$x=5,$$
$$y=2x+8=10+8=18.$$

The fraction is therefore, $\frac{5}{18}$.

2. Find two numbers the greater of which shall be to the less, as their sum is to 42, and as their difference is to 6.

Let x and y be the numbers;

then, $x : y = x + y : 42,$

and, $x : y = x - y : 6;$

and since the first two terms of these proportions are the same, the last terms will be proportional; whence we have,

$$x + y : x - y = 42 : 6,$$

or, $x + y : x - y = 7 : 1.$

Comparing the sum of antecedents and consequents with their difference (Arith. 69), we have,

$$(x + y) + (x - y) : (x + y) - (x - y) = 8 : 6,$$

or, $2x : 2y = 8 : 6;$

dividing by 2, $x : y = 4 : 3;$

whence, $x = \frac{4y}{3}.$

Substituting this value of x in the proportion

$x : y = x - y : 6$, we have,

$$\frac{4y}{3} : y = \frac{4y}{3} - y : 6,$$

or (Arith. 68), $4y : 3y = 4y - 3y : 18,$

or, $3 : 3 = y : 18;$

whence, $3y = 72,$

and, $y = 24.$

$$x = \frac{4y}{3} = \frac{96}{3} = 32.$$

3. A farmer with 28 bushels of barley, at 56 cents a bushel, would mix rye at 72 cents per bushel, and wheat at 96 cents per bushel, so that the whole mixture may consist of 100 bushels, and be worth 80 cents per bushel. How many bushels of each kind must he mix with the barley?

Let $x =$ the number of bushels of rye;

And $y =$ the number of bushels of wheat;

then, $x + y + 28 = 100,$

or, $x + y = 72.$

The value of the barley at 56 cts. the bushel is, $56 \times 28 = 1568$; the rye at 72 cents, is $72x$, and the wheat at 96 cents, is $96y$; the value of the whole at 80 cents, is $100 \times 80 = 8000$;

wherefore, $1568 + 72x + 96y = 8000.$

Transposing, $72x + 96y = 6432.$

Dividing by 24, $3x + 4y = 268.$

Multiplying the first equation by 3, we have,

$$3x + 3y = 216;$$

and subtracting it from the last,

$$y = 52.$$

$$x = 72 - y = 72 - 52 = 20.$$

4. The dimensions of a certain rectangular floor are such, that had it been 4 feet longer, and 2 feet wider, it would have been 60 square feet larger; and had it been 2 feet longer and 4 feet wider, it would have been 64 square feet larger. Required the dimensions.

Let $x =$ the length.

And $y =$ the breadth.

Then xy, the length multiplied into the breadth, is the number of square feet it contains. Now, had it been 4 feet longer, the breadth would have been $x + 4$; and had it been 2 feet broader, the breadth would have been $y + 2$; we have, therefore, $(x + 4) \times (y + 2) = xy + 60;$

and for a like reason,

$$(x + 2) \times (y + 4) = xy + 64.$$

Performing the multiplications indicated, these equations become,

$$\left\{ \begin{array}{l} xy + 4y + 2x + 8 = xy + 60 \\ xy + 2y + 4x + 8 = xy + 64 \end{array} \right\}$$

or, cancelling xy, which is found in both members of the two equations,

$$\left\{ \begin{array}{l} 4y + 2x + 8 = 60 \\ 2y + 4x + 8 = 64 \end{array} \right\}$$

Dividing the second equation by 2, and subtracting it from the first, there remains,

$$3y - 4 = 28;$$

hence, $3y = 24,$

$$y = 8;$$

and $x = \dfrac{60 - 8 - 4y}{2} = \dfrac{52 - 32}{2} = 10.$

5. A, B and C set down to play, each one with a certain number of shillings; A loses to B and C as many shill

as each of them has. Next B loses to A and C as many as each of them now has. Lastly, C loses to A and B as many as each of them now has. After all, each one of them has 16 shillings. How much did each one gain or lose?

Let $x =$ A's number of shillings at the first;
$y =$ B's;
$z =$ C's.

Then since B and C double their money at the first game, and A loses what they gain; at the end of this game

A will have $x - y - z$,
B will have $2y$,
C will have $2z$.

At the end of the second game, since A and C double their money, and A loses what they gain,

A has $2x - 2y - 2z$,
B has $2y - (x - y - z) - 2z$, or $3y - x - z$,
C has $4z$.

At the end of the third game, A and B have each double what they had at the beginning of this game, and all they gain, C loses. Wherefore, at the end of the third game,

A has $4x - 4y - 4z$,
B has $6y - 2x - 2z$,
C has $4z - (2x - 2y - 2z) - (3y - x - z)$,
or $7z - x - y$.

But by the conditions of the question, each has 16 shillings at the end of the third game; we have, therefore, the three equations,

$$4x - 4y - 4z = 16,$$
$$6y - 2x - 2z = 16,$$
$$7z - x - y = 16.$$

Reducing these equations, according to the preceding rules, we shall find

$$x = 26, \ y = 14, \ z = 8.$$

Therefore A lost 10 shillings, B gained 2, and C 8.

6. If A and B together, can perform a piece of work in 8 days, A and C together, in 9 days, and B and C, in 10 days, how many days will it take each person alone, to perform the same work?

Let x, y, and z be the number of days in which A, B, and C respectively, would perform the work alone; then A would perform in one day a part represented by $\frac{1}{x}$, B would perform in one day, a part $\frac{1}{y}$, and C a part $\frac{1}{z}$. Now, A and B together, will perform in one day one-eighth part of it, since they perform the whole in 8 days. We have, therefore,

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{8};$$

and for a like reason,

$$\frac{1}{x} + \frac{1}{z} = \frac{1}{9};$$

and,

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{10}.$$

Subtracting the second of these equations from the first, we have,

$$\frac{1}{y} - \frac{1}{z} = \frac{1}{8} - \frac{1}{9} = \frac{1}{72}.$$

Adding this last equation to the third equation, we obtain

$$\frac{2}{y} = \frac{1}{72} + \frac{1}{10} = \frac{82}{720}.$$

Therefore, $$82y = 1440,$$

and, $$y = 17\tfrac{23}{41}.$$

To find the value of x, let the third equation be subtracted from the first, there results the equation,

$$\frac{1}{x} - \frac{1}{z} = \frac{1}{8} - \frac{1}{10} = \frac{1}{40}.$$

Adding to this the second equation, we have,

$$\frac{2}{x} = \frac{1}{9} + \frac{1}{40} = \frac{49}{360};$$

whence, $$49x = 720,$$

and, $$x = 14\tfrac{34}{49}.$$

To find z, subtract the equation $\frac{1}{x} - \frac{1}{z} = \frac{1}{40}$ from the second equation, and there results the equation,

$$\frac{2}{z} = \frac{1}{9} - \frac{1}{40} = \frac{31}{360};$$

whence, $31z = 720$,

and, $z = 23\frac{7}{31}$.

Therefore A would perform the work in $14\frac{24}{44}$ days, B in $17\frac{23}{41}$, and C in $23\frac{7}{31}$.

7. The sum of two numbers is 12, and the difference of their squares 48. Required the numbers.

Let x and y be the numbers, then

$$x + y = 12,$$

and, $x^2 - y^2 = 48$.

Dividing the second equation by the first, term by term, we have, $x - y = 4$;

and adding to this the first equation,

$$2x = 16,$$

$$x = 8,$$

and, $y = x - 4 = 8 - 4 = 4$.

8. Find two numbers, such that half the first, with a third part of the second, make 9, and a fourth part of the first, with a fifth part of the second, make 5.

Ans. 8 and 15.

9. A gentleman being asked the age of his two sons, answered, that if to the sum of their ages 18 be added, the result will be double the age of the elder; but if 6 be taken from the difference of their ages, the remainder will be equal to the age of the younger. What were their ages?

Ans. 30 and 12.

10. There is a number consisting of 3 digits, the sum of which is 7, the right hand digit is one third of the left; and if 198 be subtracted from the number itself, the digits will be inverted. Required the number. Ans. 331.

11. There are two numbers, such that $\frac{1}{2}$ the greater added to $\frac{1}{3}$ the lesser, is 13; and if $\frac{1}{2}$ the lesser is taken from $\frac{1}{3}$ the greater, the remainder is nothing. Required the numbers.

Ans. 18 and 12.

12. A gentleman left a sum of money to be divided among four servants, so that the share of the first was $\frac{1}{2}$ the sum of the shares of the other three; the share of the second, $\frac{1}{3}$ of the sum of the other three; and the share of the third, $\frac{1}{4}$ the sum of the other three; and it was also found that the

share of the last was 14 dollars less than that of the first. What was the amount of money divided, and the shares of each respectively?

Ans. The sum was $120; the shares 40, 30, 24, and 26.

13. What two numbers are those whose difference, sum, and product, are to each other, as the numbers 2, 3, and 5 respectively. Ans. 10 and 2.

14. A person exchanged 12 bushels of wheat for 8 bushels of barley and £2 16s., offering at the same time to exchange a certain quantity of wheat for an equal quantity of barley and £3 15s., or to sell it for £10 in money. Required the price of wheat and barley per bushel.

Ans. 8 shillings, and 5 shillings, respectively.

15. Two persons, A and B, can perform a piece of work in 16 days. They work together for 4 days, when A being called off, B is left to finish it, which he does in 26 days. In what time would each do it separately?

Ans. A in 24 days; B in 48 days.

16. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$. Ans. $\frac{4}{5}$.

17. A corn factor mixes wheat flour, which cost him 10 shillings per bushel, with barley flour, which costs 4 shillings per bushel, in such proportion as to gain $43\frac{3}{4}$ per cent. by selling the mixture at 11 shillings per bushel. Required the proportion.

Ans. The proportion is 14 bushels of wheat flour to 9 of barley.

18. Find three numbers, such that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third shall be equal to 62; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third equal to 47; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third equal to 38.

Ans. 24, 60, and 120.

19. A, B and C loan different sums of money at different rates of interest. B loans 300 dollars more than A, at a rate one per cent. less, and receives 2 dollars more interest; C loans

200 dollars less than A, at one per cent. more, and receives 2 dollars less interest. How much did each loan, and at what rate?

Ans. A loaned 1000 dollars at 5 per cent., B 1300 at 4 per cent., and C 800 at 6 per cent.

20. \$500 was to be lent out at interest in two separate sums, the smaller at 2 per cent. more than the other. The interest of the greater sum was afterwards increased, and that of the smaller sum diminished by one per cent. By this the interest of the whole was augmented by one-fourth of the former value. But if the interest of the greater sum had been so increased without any diminution of the less, the interest of the whole would have been increased one-third. What were the sums and the rate per cent. of each?

Ans. The sums were \$100 and \$400, and the rates per cent. 4, and 2, respectively.

CHAPTER III.

INVOLUTION AND EVOLUTION OF ALGEBRAIC QUANTITIES.

INVOLUTION.

(38.) Involution is the raising of a quantity to any proposed power. It may be performed, as in arithmetic, by multiplying the given quantity by itself a number of times less *one* than the degree of the power; thus, the third power of $3a^2b$ is,

$$3a^2b \times 3a^2b \times 3a^2b = 27a^6b^3.$$

But, in performing this operation, the coefficient is raised to the given power, and the exponents of the letters added for each time the multiplication is performed. In this example the exponents are taken three times; and it is manifest that the same result would have been obtained, had the coefficients been raised to the required power, and the exponents multiplied by 3, the index of the power.

RULE.

Raise the coefficient of the given quantity to the required power, and multiply the exponents of the letters by the index of the power.

EXAMPLES.

1. What is the 5th power of $2a^2bc^2$?

The 5th power of 2 is 32; and the exponents of the letters being multiplied by 5 respectively, give a^{10}, b^5, and c^{10}; and hence, $32a^{10}b^5c^{10}$ is the required power.

NOTE.—All the even powers of a negative quantity will be positive, and the odd powers negative, for we have,

$$-a \times -a = a^2, \text{ positive.}$$
$$a^2 \times -a = -a^3, \text{ negative.}$$
$$-a^3 \times -a = a^4, \text{ positive.}$$
$$-a^4 \times -a = -a^5, \text{ negative, \&c.}$$

2. Required the 2d power of $13ab^2$. Ans. $169a^2b^4$.

3. Required the 4th power of $-7a^3x^2$. Ans. $2401a^{12}x^8$.

4. Required the 3d power of $\frac{2}{3}abc$. Ans. $\frac{8}{27}a^3b^3c^3$.

5. Required the 2d power of $\frac{2a^2b^3}{5c}$. Ans. $\frac{4a^4b^6}{25c^2}$.

6. Required the 6th power of $-\frac{2a}{3x}$. Ans. $\frac{64a^6}{729x^6}$.

7. Required the 6th power of $\frac{a^2b}{\frac{1}{3}x}$. Ans. $\frac{729a^{12}b^6}{x^6}$.

8. Required the 7th power of $-\frac{ac^2x}{3b}$. Ans. $-\frac{a^7c^{14}x^7}{2187b^7}$.

OF NEGATIVE EXPONENTS.

(39.) The rule for the division of algebraic quantities requires, that we subtract the exponent of a letter of the di sor from the exponent of the same letter in the dividend, order to arrive at the quotient. Thus, $a^3 \div a$ gives a^2 fo quotient. If it were required to divide a^2 by a^2

the exponents being in both cases equal, would give for a result a^0. We have, therefore, $a^0 = 1$, for $\frac{a^2}{a^2} = 1$, and also $\frac{a}{a} = 1$; and in general, $\frac{a^m}{a^m} = a^0$, and also $= 1$, since any quantity is contained once in itself.

In case the exponent of the divisor is greater than that of the dividend, there results a negative exponent, for $a^2 \div a^3 = a^{2-3} = a^{-1}$. In like manner, we have, $a \div a^3$, or $\frac{a}{a^3} = a^{-2}$, $\frac{a}{a^4} = a^{-3}$, and in general, $\frac{a}{a^{m+1}} = a^{-m}$. Now, $\frac{a}{a^2}$, $\frac{a}{a^3}$, $\frac{a}{a^4}$ and $\frac{a}{a^{m+1}}$, may be reduced by dividing both terms of each fraction by a, and we have,

$$\frac{a}{a^2} = \frac{1}{a},\ \frac{a}{a^3} = \frac{1}{a^2},\ \frac{a}{a^4} = \frac{1}{a^3},\ \text{and}\ \frac{a}{a^{m+1}} = \frac{1}{a^m};$$

wherefore,

$$a^{-1} = \frac{1}{a},\ a^{-2} = \frac{1}{a^2},\ a^{-3} = \frac{1}{a^3},\ \text{and}\ a^{-m} = \frac{1}{a^m};$$

and hence we conclude, *that a quantity affected with a negative exponent, is equal to unity divided by the same quantity raised to a power denoted by that exponent.*

This being understood, we can perform upon quantities affected with negative exponents, the same operations as upon those affected with positive exponents, and the same rules will also apply. Thus,

$a^{-2} \times a^{-3} = a^{-5}$ for $\frac{1}{a^2} \times \frac{1}{a^3} = \frac{1}{a^5}$, which is the same as a^{-5}; also, $a^{-1} \times a^3 = a^2$ for $\frac{1}{a} \times a^3 = \frac{a^3}{a} = a^2$. We have, likewise, $a^{-3} \div a^{-2} = a^{-3+2} = a^{-1}$; for $\frac{1}{a^3} \div \frac{1}{a^2} = \frac{1}{a} = a^{-1}$.

So also, if we would raise a quantity having a negative exponent to any given power, we multiply the exponent by the index of the power; thus, $(2a^{-2})^2 = 4a^{-4}$; for, $2a^{-2}$ is the same as $2\frac{1}{a^2}$, the square of which is $4\frac{1}{a^4}$, or $\frac{4}{a^4} = 4a^{-4}$.

EXAMPLES.

1. Multiply a^{-2} by a^{-4}. Ans. a^{-6}, or $\frac{1}{a^6}$.
2. Multiply $2a^2b^{-1}$ by $5a^{-1}b^2$. Ans. $10ab$.
3. Multiply $16a^3b^{-1}$ by $7a^{-3}b^2$. Ans. $112a^0b$, or $112b$.
4. Divide $9a^{-2}$ by $3a^{-1}$. Ans. $3a^{-1}$, or $\frac{3}{a}$.
5. Divide $36a^2b^{-2}$ by $9a^{-2}b^2$. Ans. $4a^4b^{-4}$, or $\frac{4a^4}{b^4}$.
6. Divide $33a^2$ by $11a^{-3}$. Ans. $3a^5$.
7. Required the 3d power of $6a^{-2}b^{-1}$. Ans. $216a^{-6}b^{-3}$.
8. Required the 2d power of $11a^{-1}b^2$. Ans. $121a^{-2}b^4$.
9. Required the 5th power of $2a^{-1}x^{-2}$. Ans. $32a^{-5}x^{-10}$, or $\frac{32}{a^5x^{10}}$.

INVOLUTION OF POLYNOMIALS.

RULE.

(40.) *Multiply the polynomial by itself a number of times less* one *than the degree of the required power.*

EXAMPLES.

1. Required the 2d, 3d, 4th, 5th, and 6th powers of $a+b$.

$$\begin{array}{l}
a+b \\
a+b \\
\hline
a^2+ab \\
\quad +ab+b^2 \\
\hline
a^2+2ab+b^2 \quad \text{2d power.} \\
a+b \\
\hline
a^3+2a^2b+ab^2 \\
\quad + \ a^2b+2ab^2+b^3 \\
\hline
a^3+3a^2b+3ab^2+b^3 \quad \text{3d power.} \\
a+b \\
\hline
a^4+3a^3b+3a^2b^2+ab^3 \\
\qquad\quad a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
a^4+4a^3b+6a^2b^2+4ab^3+b^4 \quad \text{4th power.} \\
a+b \\
\hline
a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
\qquad\quad a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\
\hline
a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \quad \text{5th power.} \\
a+b \\
\hline
a^6+5a^5b+10a^4b^2+10a^3b^3+5a^2b^4+ab^5 \\
\qquad\quad a^5b+5a^4b^2+10a^3b^3+10a^2b^4+5ab^5+b^6 \\
\hline
a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6 \ \text{6th power.}
\end{array}$$

2. Required the 2d and 3d powers of $3a^2b - 2ab^2$.

$$\begin{array}{l}
3a^2b - 2ab^2\\
3a^2b - 2ab^2\\
\hline
9a^4b^2 - 6a^3b^3\\
\qquad\quad - 6a^3b^3 + 4a^2b^4\\
\hline
9a^4b^2 - 12a^3b^3 + 4a^2b^4 \quad \text{2d power.}\\
3a^2b - 2ab^2\\
\hline
27a^6b^3 - 36a^5b^4 + 12a^4b^5\\
\qquad\quad - 18a^5b^4 + 24a^4b^5 - 8a^3b^6\\
\hline
27a^6b^3 - 54a^5b^4 + 36a^4b^5 - 8a^3b^6 \quad \text{3d power.}
\end{array}$$

(41.) In the same manner, the different powers of any polynomial may be obtained; but as the method is somewhat tedious, we proceed to investigate the laws according to which the powers of polynomials may be obtained without the actual multiplication of the polynomial by itself. We begin with the 2d power or square. We have already seen (40), that

$$(a+b)^2 = a^2 + 2ab + b^2,$$

and, $$(a-b)^2 = a^2 - 2ab + b^2.$$

If, now, it were required to form the 2d power of $2a^2b^2 + 3a$, we may put $2a^2b^2 = a'$, and $3a = b'$, and we shall have,

$$(a'+b')^2 = a'^2 + 2a'b' + b'^2.$$

If, in this expression, we put in the place of a' and b' their values, we shall have,

$$(2a^2b^2 + 3a)^2 = (2a^2b^2)^2 + 2(2a^2b^2 \times 3a) + (3a)^2;$$

and performing the operations indicated,

$$(2a^2b^2 + 3a)^2 = 4a^4b^4 + 12a^3b^2 + 9a^2.$$

By a process entirely similar, we should find,

$$(2a^2b^2 - 3a)^2 = 4a^4b^4 - 12a^3b^2 + 9a^2.$$

We hence conclude, that *the square of any binomial is found by taking the square of the first term, twice the product of the two terms, and the square of the last term, and connecting them by their proper signs.*

If the second term of the binomial be negative, the double product will be negative, since it results from the multiplication of quantities having unlike signs.

Suppose it required to find the second power of $a+b+c$. Making $a+b=a'$, we have,

$$a+b+c=a'+c,$$

and, $$(a'+c)^2=a'^2+2a'c+c^2.$$

By substituting, in this expression, $a+b$, in the place of a', we have,

$$(a+b+c)^2=(a+b)^2+2(a+b)c+c^2,$$

which becomes, by performing the operations indicated,

$$(a+b+c)^2=a^2+2ab+b^2+2ac+2bc+c^2.$$

By inspecting this result, it will be seen that the square of a polynomial of three terms is composed of the square of the first term, twice the product of the first by the second, the square of the second, twice the product of the first by the third, twice the product of the second by the third, and the square of the third. Hence, *to raise a polynomial of three terms to the second power, we have only to take the squares of the three terms, and the double products of these terms taken two and two, and connect them by their proper signs.*

The squares of the terms will all have the sign $+$, and for the double products, the rules for the signs in multiplication must be applied; thus,

$$(5a-2ab+2c^2)^2=25a^2-20a^2b+4a^2b^2+20ac^2$$
$$-8abc^2+4c^4.$$

To find the square of a polynomial of four terms, as $a+b+c+d$, we make $a'=a+b+c$, and the given polynomial becomes, $a'+d$, and $(a'+d)^2=a'^2+2a'd+d^2$. Putting for a', in this expression, its value $a+b+c$, we have,

$$(a+b+c+d)^2=(a+b+c)^2+2(a+b+c)d+d^2.$$

Squaring $a+b+c$ by the last rule, and performing the multiplication indicated, we obtain,

$$(a+b+c+d)^2=a^2+2ab+b^2+2ac+2bc+c^2+2ad+2bd$$
$$+2cd+d^2.$$

From this result, it will be perceived, that *the square of a polynomial of four terms, is composed of the squares of all its terms, and the double products of the terms taken two and two.*

We now propose to show that this rule for the formation of the squares of polynomials is general, or, that the square of any polynomial is composed of the squares of all its terms, and their double products taken two and two.

Let $a+b+c+d \ldots . +f$ be a polynomial having m terms; then the polynomial $a+b+c+d \ldots . +f+h$ has $m+1$ terms. We say now, that if the square of the first polynomial is composed of the squares of all the terms, and their double products taken two and two, the square of the second will follow the same rule.

Let $a+b+c+d \ldots . +f=a'$; then, $a+b+c+d \ldots . +f+h=a'+h$. Now the square of $a'+h$ is $a'^2+2a'h+h^2$, and if we put in the place of a', its value, we shall have,

$$(a+b+c+d \ldots . +f+h)^2=(a+b+c+d \ldots . +f)^2 \\ +2(a+b+c+d \ldots . +f)\times h+h^2.$$

By hypothesis, the first part of this expression is equal to the squares of all its terms, and their double products taken two and two; the second part is the double products of all these terms by the new term h, and the third is the square of this term; hence, if the rule is true for a polynomial of m terms, it is true for one of $m+1$ terms.

We have proved that it is true for a polynomial of four terms, it is therefore true for one of five terms; and if true for one of five terms, it is for one of six, and so on for any number of terms.

For finding the second power of polynomials, we have, therefore, the following

RULE.

Raise each term to the second power, and multiply twice the term itself into all the terms that precede it.

EXAMPLES.

1. $(a+b+c)^2=a^2+2ab+b^2+2ac+2bc+c^2$.

2. $(2a^2+2x+x^2)^2=4a^4+8a^2x+4x^2+4a^2x^2+4x^3+x^4$.

3. $(5a-x^2-bx)^2=25a^2-10ax^2+x^4-10abx+2bx^3$
$+b^2x^2$.

4. $\left(\frac{2a^2-3ax+x^2}{2a-x}\right)^2=\frac{4a^4-12a^3x+13a^2 \qquad x^4}{4a^2-4a}$.

5. What is the second power of $\frac{ax+ac-2cx}{a}$.

Ans. $\frac{a^2x^2+2a^2cx+a^2c^2-4acx^2-4ac^2x+4c^2x^2}{a^2}$.

6. What is the second power of $a^2-2ab+b^2$?

Ans. $a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

7. What is the second power of $a^3+3a^2b+3ab^2+b^3$?

Ans. $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+5ab^5+b^6$.

8. What is the second power of $4a+2b-b^2$?

Ans. $16a^2+16ab+4b^2-8ab^2-4b^3+b^4$.

9. What is the second power of $7a^2-4ax+3x^2$?

Ans. $49a^4-56a^3x+58a^2x^2-24ax^3+9x^4$.

10. What is the second power of $a^2c^2-2abcx+b^2x^2$?

Ans. $a^4c^4-4a^3bc^3x+6a^2b^2c^2x^2-4ab^3cx^3+b^4x^4$.

(42.) The third powers of polynomials may be found by a method similar to that used for finding the square. We find by actual multiplication (40),

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3.$$

By inspecting this result, it will be seen that the third power of $a+b$ consists of four parts, namely; the third power of a, three times the square of a, multiplied by b, three times a multiplied by the square of b, and the third power of b; and, as any quantities whatever can be substituted in the place of a and b, we can find the third power of any binomial by the following

Rule.

To the third power of the first term, add three times the square of the first multiplied by the second, three times the first multiplied by the square of the second, and the third power of the second.

Examples.

1. $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$.

Note.—The signs with which the different terms must be affected are determined by applying the rules for multiplication; thus, in the example above, the second term has the sign —, for $3a^2 \times - b = - 3ab$; the third term has the sign +, since $3a \times (- b)^2$, or $3a \times b^2 = 3ab$; and the last being an odd power of a negative quantity, has the sign — (38).

2. $(2a^2+ax)^3 = (2a^2)^3+3(2a^2)^2 (ax)+3(2a^2) (ax)^2+(ax)^3$
$= 8a^6 + 12a^5x + 6a^4x^2 + a^3x^3$.

3. $(5a - 10x)^3 = 125a^3 - 750a^2x + 1500ax^2 - 1000x^3$.

4. $(a^2x^2 + 2ax^3)^3 = a^6x^6 + 6a^5x^7 + 12a^4x^8 + 8a^3x^9$.

5. $(2ab - 2bc)^3 = 8a^3b^3 - 24a^2b^3c + 24ab^3c^2 - 8b^3c^3$.

6. $\left(\frac{3a + 6b}{2a^2}\right)^3 = \frac{27a^3 + 162a^2b + 324ab^2 + 216b^3}{8a^6}$.

If it were required to find the third power of a polynomial of three terms, as $a + b + c$, we make $a + b = a'$. We then have,

$$(a' + c)^3 = a'^3 + 3a'^2c + 3a'c^2 + c^3;$$

and substituting in the place of a', its value $a + b$, we have,

$$(a+b+c)^3 = (a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$$
$$= a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3.$$

From this result it appears, that *the third power of a trinomial is composed of the third powers of its three terms, three times the square of each term, multiplied by each of the other terms respectively, and six times the product of all the terms.*

EXAMPLES.

1. $(2a^2 - ax+x^2)^3 = 8a^6 - 12a^5x+6a^4x^2 - a^3x^3+12a^4x^2$
$+6a^2x^4+3a^2x^4-3ax^5-12a^3x^3+x^6$
$=8a^6-12a^5x+18a^4x^2-13a^3x^3+9a^2x^4-3ax^5+x^6$.

2. $(a^2 - 2ax + x^2)^3 = a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4$
$-6ax^5+x^6$.

3. $(10a+3ax-x^2)^3 = 1000a^3+900a^3x+270a^3x^2+27a^3x^3$
$-300a^2x^2-180a^2x^3-27a^2x^4+30ax^4+9ax^5-x^6$.

If it be required to find the third power of a polynomial having four terms as, $a + b + c + d$, we have only to make $a + b = a'$, and $c + d = b'$, and after having raised $a' + b'$ to the third power, substitute in the place of a' and b', their values; thus,

$$(a + b + c+d)^3 = (a+b)^3 + 3(a+b)^2(c+d) + 3(a+b)(c+d)^2$$
$$+ (c + d)^3.$$

In general, if it be required to raise a polynomial to any power whatever, it can always be done by making its terms equal to $a' + b'$, and afterwards substituting these terms in the developement of $a' + b'$.

For example; let $3a + bc$ be raised to the fourth power. Making $3a = a'$, and $bc = b'$, we have (40),

$$(a' + b')^4 = a'^4 + 4a'^3b' + 6a'^2b'^2 + 4a'b'^3 + b'^4.$$

Putting in the place a' and b', their values, this expression becomes,

$$(3a^2+bc)^4 = (3a^2)^4 + 4(3a^2)^3(bc) + 6(3a^2)^2(bc)^2 + 4(3a^2)(bc)^3 + (bc)^4$$
$$= 81a^8+108a^6bc+54a^4b^2c^2+12a^2b^3c^3+b^4c^4.$$

In like manner we find,

$$(5a - 2cx^2)^4 = (5a)^4 - 4(5a)^3(2cx^2) + 6(5a)^2(2cx^2)^2 - 4(5a)(2cx^2)^3 + (2cx^2)^4$$
$$= 625a^4 - 1000a^3cx^2 + 600a^2c^2x^4 - 160ac^3x^6 + 8c^4x^8.$$

(43.) It will be seen, by reference to Art. (40), that

$$(a + b)^2 = a^2 + 2ab + b^2,$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4, \text{ \&c.}$$

These powers of $a + b$ were obtained by actual multiplication; but any power of $a + b$ may be found much more easily by the following

RULE.

1. *Write, for the first term,* a *with an exponent equal to the required power; for the second,* a *with an exponent one less, multiplied by* b; *and for the remaining terms, diminish the exponent of* a *by unity, and increase that of* b *by unity, for each term, proceeding from left to right, till the exponent of* a *becomes* 0, *and the exponent of* b *equal to the required power.*

2. *The coefficient of the first term is unity; the coefficients of the other terms are found by the following rule: Multiply the coefficient of any term by the exponent of* a *in that term, and divide by the number of that term, counting from the left. The quotient will be the coefficient of the next term.*

Let it be required to find the eighth power of $(a+b)$, we shall have by the rule,

$$(a+b)^8=a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8.$$

In this example, the first term is x^8, the required power of the binomial; the second term has a^7, the third a^6, and so on, the exponent of a being one less in each term as we proceed towards the right, till we arrive at the eighth term, where it has unity for an exponent. The powers of b, commencing at the second term, increase to the last, in which it has an exponent equal to the required power. It will also be perceived, that by adding together the exponents of a and b in any term, the sum is always equal to 8, the required power of the binomial.

As respects the coefficients, we have unity for the coefficient of the first term, and multiplying this by 8, the exponent of a in that term, and dividing by 1, the number of the term, gives 8 for the coefficient of the second term; multiplying this by 7, the exponent of a, and dividing by 2, the number of the term, gives $\frac{8\times7}{2}=28$, for the coefficient of the third term. Proceeding in like manner with the rest, we obtain for the coefficient of the fourth term, $\frac{28\times6}{3}=56$; for the coefficient of the fifth term, $\frac{56\times5}{4}=70$; for the sixth, $\frac{70\times4}{5}=56$; for the seventh, $\frac{56\times3}{6}=28$; for the eighth, $\frac{28\times2}{7}=8$; and for the coefficient of the last term, $\frac{8\times1}{8}=1$.

Examples.

1. Find the seventh power of $(a-b)$.

$$(a-b)^7=a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7.$$

In this example, the signs of the terms are alternately + and —, which is always the case in any power of $a-b$. The reason for this is obvious, since the terms have alternately the odd and even powers of $-b$; all the odd powers of $-b$ must be preceded by the sign —, and the even ones by the sign + (38); and hence, these powers of $-b$, multiplied into the powers of a, all of which have the sign +, will give alternately + and —.

2. Find the mth power of $(a+b)$.

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{2}a^{m-2}b$$
$$+\frac{m(m-1)(m-2)}{2.3}a^{m-3}b^3 + \frac{m(m-1)(m-2)(m-3)}{2.3.4.}a^{m-4}b^4$$
$$+\frac{m(m-1)(m-2)(m-3)(m-4)}{2.3.4.5}a^{m-5}b^5 + \&c.$$

In an example like the preceding, the series will not terminate, unless a numerical value is given to m. If we make $m=4$, it terminates at the fifth term; for in that case $a^{m-4}=a^0$. The coefficient of the fifth term then becomes unity, for

$$\frac{4(4-1)(4-2)(4-3)}{2.3.4}=\frac{4\,3\,2.1}{2.3.4}=\frac{24}{24}=1,$$

and the fifth term is a^0b^4 or b^4. In like manner, if we make $m=3$, the series terminates at the fourth term; and if $m=5$, it terminates at the sixth term.

3. Find the 6th power of $a-x$.

$$(a-x)^6=a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6.$$

4. Find the 5th power of $2a^2+3x$.

$$(2a^2+3x)^5 = (2a^2)^5+5(2a^2)^4 3x+10(2a^2)^3(3x)^2+10(2a^2)^2(3x)^3$$
$$+5(2a^2)(3x)^4+(3x)^5$$
$$= 32a^{10}+240a^8x+720a^6x^2+1080a^4x^3+810a^2x^4+243x^5.$$

5. $(4ac^2-5c^3)^4 = (4ac^2)^4 - 4(4ac^2)^3(5c^3) + 6(4ac^2)^2(5c^3)^2$
$$-4(4ac^2)(5c^3)^3+(5c^3)^4$$
$$=256a^4c^8-1280a^3c^9+2400a^2c^{10}-2000ac^{11}+625c^{12}.$$

6. $(4a+3x^3)^6 = (4a)^6 + 6(4a)^5(3x^3) + 15(4a)^4(3x^3)^2$
$+20(4a)^3(3x^3)^3 + 15(4a)^2(3x^3)^4 + 6(4a)(3x^3)^5+(3x^3)^6.$
$$=4096a^6+18432a^5x^3+34560a^4x^6+34560a^3x^9+19440a^2x^{12}$$
$$+5832ax^{15}+729x^{18}.$$

7. $(9a^2-4b^2)^4=6561a^8-11664a^6b^2+7776a^4b^4-2304a^2b^6$
$$+256b^8.$$

EVOLUTION.

(44.) Evolution is the reverse of involution. It is the finding or extracting the root of a given quantity.

The *root* of a quantity is that quantity which, raised to a power denoted by the index of the root, shall be equal to the given quantity. Thus, the square root of a quantity, raised to the second power is equal to that quantity. So also the 3d, 4th, 5th and 6th roots of a quantity, raised to the 3d, 4th, 5th and 6th powers, respectively, will be equal to that quantity.

The signs of the roots in algebra are the same as in arithmetic. $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, $\sqrt[5]{a}$, $\sqrt[6]{a}$, &c. indicate the square root, the cube root, the 4th root, the 5th root, and the 6th root of a, respectively. The figure before the sign shows the degree of the root, and is called the index of the root.

Extraction of the Roots of Monomials.

Rule.

Extract the root of the coefficient, and divide the exponent of each letter by the index of the root.

The reason for this rule is evident; for since a monomial is raised to any power by raising the coefficient to that power, and multiplying the exponent of each letter by the degree of the power, the root of that power will be obtained by reversing the process; that is, by extracting the root of the coefficient, and *dividing* the exponent of each letter by the index of the root. Thus, the cube root of $125a^3b^6$ is $5ab^2$; for $(5ab^2)^3 = 125a^3b^6$.

Examples.

1. $\sqrt{9a^4b^2} = 3a^2b$.
2. $\sqrt[3]{27a^3b^6} = 3ab^2$.
3. $\sqrt[4]{16a^8x^{12}} = 2a^2x^3$.
4. $\sqrt[5]{243b^5c^{10}} = 3bc^2$.
5. $\sqrt[6]{64a^{12}x^6} = 2a^2x$.

6. $\sqrt{16a^{-4}b^{2}} = 4a^{-2}b$, or $\frac{4b}{a^{2}}$.

7. $\sqrt[3]{8a^{-3}b^{-6}c^{3}} = 2a^{-1}b^{-2}c$, or $\frac{2c}{ab^{2}}$.

8. $\sqrt{\frac{9a^{6}b^{4}}{36c^{4}}} = \frac{3a^{3}b^{2}}{6c^{2}}$.

9. $\sqrt[4]{81b^{-4}c^{4}x^{-8}}$, or $\sqrt[4]{\frac{81c^{4}}{b^{4}x^{8}}} = 3b^{-1}cx^{-2}$, or $\frac{3c}{bx^{2}}$.

10. $\sqrt[n]{a^{n}x^{2n}y^{3n}} = ax^{2}y^{3}$.

(45.) When the exponents of the letters are not divisible by the index of the root, the root cannot be extracted; but in that case algebraists have agreed to indicate the extraction by writing the index of the root beneath the exponent of the letter. Thus, if it were required to extract the square root of a^{3}, we should indicate the operation by writing 2 beneath the exponent in a fractional form, and we should have $\sqrt{a^{3}} = a^{\frac{3}{2}}$. So also, $\sqrt[n]{a^{m}} = a^{\frac{m}{n}}$. These are termed fractional exponents and may be considered expressions for roots which cannot be extracted.

Examples.

1. $\sqrt{a^{3}b^{2}} = a^{\frac{3}{2}}b$.

2. $\sqrt[3]{a^{2}b^{4}c} = a^{\frac{2}{3}}b^{\frac{4}{3}}c^{\frac{1}{3}}$.

3. $\sqrt[4]{16a^{4}bc^{2}} = 2ab^{\frac{1}{4}}c^{\frac{2}{4}}$, or $2ab^{\frac{1}{4}}c^{\frac{1}{2}}$.

4. $\sqrt{16a^{-3}b^{n}} = 4a^{-\frac{3}{2}}b^{\frac{n}{2}}$.

5. $\sqrt[n]{a^{2}b^{n}c^{3}} = a^{\frac{2}{n}}bc^{\frac{3}{n}}$.

6. $\sqrt{\frac{a^{2n}b^{2}c}{a^{m}}} = \frac{a^{n}bc^{\frac{1}{2}}}{a^{\frac{m}{2}}}$.

EXTRACTION OF THE SQUARE ROOT OF POLYNOMIALS.

RULE.

(46.) 1. *Arrange the polynomial according to the powers of a particular letter.*

2. *Find the square root of the first term, which must always be a perfect square, and place this root in the quotient. Subtract the square of this root from the first term, and there will be no remainder.*

3. *Bring down the next two terms for a dividend, and double the root for a divisor; see how often the divisor is contained in the first term of the dividend, and add the quotient to the root already found, and also to the divisor.*

4. *Multiply the divisor so increased by the aforementioned quotient, and subtract the product from the dividend. To the remainder bring down the next two terms for a new dividend, and double the two terms of the root already found for a new divisor, and proceed as before; continue the operation in the same manner until all the terms of the polynomial are brought down.*

For an example, find the square root of $16a^4+96a^3x+216a^2x^2+216ax^3+81x^4$.

$$\begin{array}{l|l}
 & \begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{The root.} \\ 16a^4+96a^3x+216a^2x^2+216ax^3+81x^4 \,\|\, 4a^2+12ax+9x^2 \\ 16a^4 \end{array} \\ \hline
\text{1st div. } 8a^2+12ax & 96a^3x+216a^2x^2 \\
 & 96a^3x+144a^2x^2 \\ \hline
\text{2d div. } 8a^2+24ax+9x^2 & 72a^2x^2+216ax^3+81x^4 \\
 & 72a^2x^2+216ax^3+81x^4 \\ \hline
\end{array}$$

In this example, we have $4a^2$, the square root of $16a^4$, for the first term of the root. Subtracting its square, which is equal to the first term, and bringing down the next two terms, we have, for a dividend, $96a^3x+216a^2x^2$; and doubling $4a^2$, we have $8a^2$ for a divisor, which divided into $96a^3x$, gives $12ax$ for a quotient; and this is placed both in the divisor and in the root. The divisor thus increased, becomes $8a^2+12ax$, which being multiplied by $12ax$, the product subtracted, and the next two terms added to the remainder, produces the second dividend. The whole of the root found being doubled, gives $8a^2+24ax$ for a second divisor. Dividing the first term of the dividend by the first term of this divisor, gives $9x^2$ for a quotient, which is also added to the divisor, and the sum multiplied by $9x^2$.

To understand the reason for this rule, we have only to recur to the square of a binomial, which we have found to be composed of the square of the first term, twice the product of the first and second, and the square of the second. The first term of the root is, therefore, the square root of the first term of the square; and since the second term of the square is twice the product of the two terms of the root, we shall evidently obtain the second term of the root, by dividing this double product by double the root already found. Thus, since $(a+b)^2 = a^2+2ab+b^2$, in extracting the root of the latter quantity, we take the square root of the first term, which is a, and having subtracted its square, there remains $2ab+b^2$. Doubling the root found, we have $2a$ for a divisor; and dividing the first term of this remainder by it, we have b for a quotient; b is then added to the divisor, so that the divisor, so increased, multiplied by b, may produce the dividend, in order to test the correctness of the work; thus, $(2a+b)b = 2ab+b^2$.

$$\begin{array}{r|l}
\multicolumn{2}{l}{\quad a^2+2ab+b^2 \parallel \overset{\text{the root.}}{a+b}} \\
\multicolumn{2}{l}{\quad a^2} \\
\hline
\text{Divisor, } 2a+b & 2ab+b^2 \\
 & 2ab+b^2 \\
\cline{2-2}
\end{array}$$

The same reasoning will apply when the root consists of three terms; for we have

$$(a+b+c)^2 = (a+b)^2+2(a+b)c+c^2.$$

Now having found the root of $(a+b)^2$, which is $a+b$, we have only to double this root and divide the next term, $2(a+b)c$ by it, to find the third term c, of the root. The operation may be thus represented.

$$\begin{array}{r|l}
\multicolumn{2}{l}{\quad (a+b)^2+2(a+b)c+c^2 \parallel \overset{\text{Root.}}{a+b+c}} \\
\multicolumn{2}{l}{\quad (a+b)^2} \\
\hline
2(a+b)+c & 2(a+b)c+c^2 \\
 & 2(a+b)c+c^2 \\
\cline{2-2}
\end{array}$$

If the square of $a+b+c$ had been developed, the operation would have been essentially the same, as will be seen by inspecting the operation.

$$\begin{array}{r|l}
 & a^2+2ab+b^2+2ac+2bc+c^2 \;\|\; a+b+c \\
 & a^2 \\
\hline
2a+b & 2ab+b^2 \\
 & 2ab+b^2 \\
\hline
2a+2b+c & +2ac+2bc+c^2 \\
 & +2ac+2bc+c^2 \\
\hline
\end{array}$$

If the root have four terms, or any number of terms whatever, the same reasoning will hold true; for we always have (41), $(a+b+c+d+e\ \ldots\ +i+k)^2=(a+b+c+d+e+\ \ldots\ i)^2+2(a+b+c+d+e+\ \ldots\ i)k+k^2$; and hence it will appear, that whatever be the number of terms in the root, we shall always find the next term by doubling the root already found for a divisor.

EXAMPLES.

1. Extract the square root of $a^6-4a^5x+8a^3x^2+4a^2x^2$.

$$\begin{array}{r|l}
 & a^6-4a^5x+8a^3x^3+4a^2x^2 \;\|\; a^3-2a^2x-2ax^2 \\
 & a^6 \\
\hline
2a^3-2a^2x & -4a^5x+8a^3x^3 \\
 & -4a^5x+4a^4x^2 \\
\hline
2a^3-4a^2x-2ax^2 & -4a^4x^2+8a^3x^3+4a^2x^2 \\
 & -4a^4x^2+8a^3x^3+4a^2x^2 \\
\hline
\end{array}$$

2. Extract the square root of $4a^6+4a^5x+\frac{5}{16}a^2x^4-\frac{1}{16}ax^5+\frac{1}{64}x^6$.

$$\begin{array}{r|l}
 & 4a^6+4a^5x+\frac{5}{16}a^2x^4-\frac{1}{16}ax^5+\frac{1}{64}x^6 \,\|\, 2a^3+a^2x-\frac{1}{4}ax^2+\frac{1}{8}x^3 \\
 & 4a^6 \\ \hline
4a^3+a^2x & 4a^5x+\frac{5}{16}a^2x^4 \\
 & 4a^5x+a^4x^2 \\ \hline
4a^3+2a^2x-\frac{1}{4}ax^2 & -a^4x^2+\frac{5}{16}a^2x^4 \\
 & -a^4x^2-\frac{1}{2}a^3x^3+\frac{1}{16}a^2x^4 \\ \hline
4a^3+2a^2x-\frac{1}{2}ax^2+\frac{1}{8}x^3 & \frac{1}{2}a^3x^3+\frac{1}{4}a^2x^4-\frac{1}{16}ax^5+\frac{1}{64}x^6 \\
 & \frac{1}{2}a^3x^3+\frac{1}{4}a^2x^4-\frac{1}{16}ax^5+\frac{1}{64}x^6 \\ \hline
 & \qquad * \;\; * \;\; *
\end{array}$$

3. Extract the square root of $1+x$.

$$\begin{array}{rl}
 & 1+x \,\|\, 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\&c. \\
 & 1 \\ \hline
2+\frac{x}{2}\Big\} & x \\
 & x+\frac{x^2}{4} \\ \hline
2+x-\frac{x^2}{4}\Big\} & -\frac{x^2}{4} \\
 & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\ \hline
2+x-\frac{x^2}{4}+\frac{x^3}{16}\Big\} & \frac{x^3}{8}-\frac{x^4}{64} \\
 & \frac{x^3}{8}+\frac{x^4}{16}-\frac{x^5}{64}+\frac{x^6}{256} \\ \hline
2+x-\frac{x^2}{4}+\frac{x^3}{16}-\frac{5x^4}{128}\Big\} & -\frac{5x^4}{64}+\frac{x^5}{64}-\frac{x^6}{256} \\
 & -\frac{5x^4}{64}+\frac{5x^5}{128}+\frac{5x^6}{256} * * * * \\ \hline
\end{array}$$

4. Extract the square root of $4x^4 - 16x^3 + 24x^2 - 16x + 4$.
Ans. $2x^2 - 4x + 2$.

5. Extract the square root of $16x^4 + 24x^3 + 89x^2 + 60x + 100$.
Ans. $4x^2 + 3x + 10$.

6. Extract the square root of $4x^4 - 16x^3 + 8x^2 + 16x + 4$.
Ans. $2x^2 - 4x - 2$.

7. Extract the square root of $4a^2b^2 - 2ab^2c + \frac{1}{4}b^2c^2 + 4a^2bc - abc^2 + a^2c^2$.
Ans. $2ab - \frac{1}{2}bc + ac$.

8. Extract the square root of $16x^4 - 16x^3 + 28x^2 - 12x + 9$.
Ans. $4x^2 - 2x + 3$.

9. Extract the square root of $16x^{2n} - 16^{2n-1} + 4x^{2n-2} + 24x^n - 12x^{n-1} + 9$.
Ans. $4x^n - 2x^{n-1} + 3$.

EXTRACTION OF THE CUBE ROOT OF POLYNOMIALS.

RULE.

(47.) 1. *Arrange the polynomial with reference to the powers of some letter. Then find the cube root of the first term, which must always admit of an exact root, and place the root, thus found, in the quotient, and subtract its cube root from the first term.*

2. *Bring down the next three terms for a dividend, and take three times the square of the root already found for a divisor. Divide the first term of the dividend by this divisor; the quotient will be another term of the root.*

3. *Complete the divisor by adding to it three times the product of this last term of the root by those which precede it, and the square of this last term. Multiply the divisor, thus increased, by the last term of the root, and subtract the product, and to the remainder bring down the next three terms, and proceed in the same manner, till all the terms are brought down, and the whole root obtained.*

The reason for this rule will be seen by a reference to the third power of $a + b$, which is $a^3 + 3a^2b + 3ab^2 + b^3$.

$$\begin{array}{r|l} & a^3 + 3a^2b + 3ab^2 + b^3 \;\|\; a + b \\ & a^3 \\ \hline 3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 \\ & 3a^2b + 3ab^2 + b^3 \end{array}$$

The first term of the root is, evidently, the root of the first term of the power; and since the second term of the power is $3a^2b$, it is evident that we shall find the second term of the root by dividing this term by $3a^2$, which is three times the square of the root already found. The two last terms of the power $3ab^2 + b^3$, or $(3ab + b^2)b$, being three times the product of the two terms of the root, and the square of the last multiplied by this last term, it will be seen why this product, and the square of the last, should be added to complete the divisor. The same reasoning will apply when the root is a trinomial; since $a + b$ can first be obtained, and then c obtained from $a + b$, in the same manner as b was obtained from a.

EXAMPLE 1.

$$
\begin{array}{r|l}
 & a^6 - 6a^5x + 15a^4x^2 - 20a^3x^3 + 15a^2x^4 - 6ax^5 + x^6 \;\|\; \overset{\text{Root.}}{a^2 - 2ax + x^2} \\
 & a^6 \\
\hline
3a^4 - 6a^3x + 4a^2x^2 & -6a^5x + 15a^4x^2 - 20a^3x^3 \\
 & -6a^5x + 12a^4x^2 - 8a^3x^3 \\
\hline
3a^4 - 12a^3x + 15a^2x^2 - 6ax^3 + x^4 & 3a^4x^2 - 12a^3x^3 + 15a^2x^4 - 6ax^5 + x^6 \\
 & 3a^4x^2 - 12a^3x^3 + 15a^2x^4 - 6ax^5 + x^6 \\
\hline
\end{array}
$$

$3(a^2)^2 = 3a^4$, first partial divisor.

$3a^2 \times -2ax = -6a^3x$, triple product of the root by $-2ax$.

$(2ax)^2 = 4a^2x^2$, square of $-2ax$.

$3a^4 - 6a^3x + 4a^2x^2$, complete divisor.

$3(a^2 - 2ax)^2 = 3a^4 - 12a^3x + 12a^2x^2$, 2d partial divisor.

$3(a^2 - 2ax)x^2 = 3a^2x^2 - 6ax^3$, triple product of the root by x^2.

$(x^2)^2 = x^4$, square of the last term.

$3a^4 - 12a^3x + 15a^2x^2 - 6ax + x^4$, complete divisor.

EXAMPLE 2.

$$\begin{array}{r|l}
 & a^6 - 3a^5 + 9a^4 - 13a^3 + 18a^2 - 12a + 8 \;\|\; \overset{\text{Root.}}{a^2 - a + 2} \\
 & a^6 \\
\hline
3a^4 - 3a^3 + a^2 & -3a^5 + 9a^4 - 13a^3 \\
 & -3a^5 + 3a^4 - \quad a^3 \\
\hline
3a^4 - 6a^3 + 3a^2 + 6a^2 - 6a + 4 & 6a^4 - 12a^3 + 18a^2 - 12a + 8 \\
 & 6a^4 - 12a^3 + \;6a^2 + 12a^2 - 12a + 8 \\
\hline
\end{array}$$

EXAMPLE 3.

$$\begin{array}{r|l}
 & \frac{1}{27}a^3b^3 - \frac{1}{6}a^2b^2x + \frac{1}{4}abx^2 - \frac{1}{8}x^3 \;\|\; \overset{\text{Root.}}{\frac{1}{3}ab - \frac{1}{2}x} \\
 & \frac{1}{27}a^3b^3 \\
\hline
\frac{1}{3}a^2b^2 - \frac{1}{2}abx + \frac{1}{4}x^2 & -\frac{1}{6}a^2b^2x + \frac{1}{4}abx^2 - \frac{1}{8}x^3 \\
 & -\frac{1}{6}a^2b^2x + \frac{1}{4}abx^2 - \frac{1}{8}x^3 \\
\hline
\end{array}$$

4. Extract the cube root of $27a^3+108a^2+144a+64$.

Ans. $3a+4$.

5. Extract the cube root of $a^3-6a^2x+12ax^2-8x^3$.

Ans. $a-2x$.

6. Extract the cube root of $x^6-3x^5+5x^3-3x-1$.

Ans. x^2-x-1.

7. Extract the cube root of $a^9-6a^8+18a^7-32a^6+36a^5-24a^4+8a^3$.

Ans. a^3-2a^2+2a.

GENERAL RULE FOR EXTRACTING THE ROOTS OF ALL POWERS.

(48.) 1. *Arrange the polynomial according to the powers of a particular letter; find the root of the first term and place it in the quotient.*

2. *Subtract its power from the first term, and there will be no remainder; then bring down the second term for a dividend.*

3. *Involve the root, last found, to the next inferior power, and multiply it by the index of the given power for a divisor. Divide the dividend by the divisor, and the quotient will be another term of the root.*

Involve the whole root, thus found, to the given power, and subtract it from the whole polynomial. Divide the first term of the remainder by the divisor first found, the quotient will be another term of the root; proceed in this manner till the whole is finished.

Extract the fourth root of $16a^4+96a^3b+216a^2b^2+216ab^3+81b^4$.

$$\begin{array}{l}
16a^4+96a^3b+216a^2b^2+216ab^3+81b^4 \;\|\; 2a+3b \\
16a^4 \\
\hline
32a^3 \,|\;\; 96a^2b+ \\
\hline
16a^4+96a^2b+216a^2b^2+216ab^3+81b^4 \\
\hline
\qquad\qquad\qquad 0
\end{array}$$

In this example, the fourth root of $16a^4$ is $2a$, which being raised to the third power, and multiplied by 4, the index of the root, gives $32a^3$ for a divisor. Dividing $96a^3b$ by this divisor, we have $3b$ for a quotient. Raising $2a+3b$ to the fourth power, and subtracting it from the given power, 0 remains, wherefore $2a+3b$ is the required root.

To prove the correctness of this rule, we have only to remark, that, in any power of a binomial $a+b$, the second term contains a in a power one less than the power itself, multiplied by b; and, that the coefficient of this second term, is always the index of the power, so that we shall have, in general, the two first terms of $(a+b)^n = a^n + na^{n-1}b$. To find the first term of the root, it is, therefore, only necessary to take the nth root of a^n; and to find the second, we have only to divide the second term of the power by na^{n-1}, or the first term of the root, raised to a power less one, and multiplied by the index of the root to be extracted. If the root consists of three terms, we may consider $(a+b)$ as a single term, and find c in the same manner as b was found in the first case.

EXAMPLES.

1. Find the cube root of $a^6-6a^5b+15a^4b^2+20a^3b^3+15a^2b^4-6ab^5+b^6$.

Root

$a^66-a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6 \,\|\, a^2-2ab+b^2$

a^6

Div.

$3a^4 \mid -6a^5b$, Dividend.

$a^6-6a^5b+12a^4b^2-8a^3b^3$

$3a^4b^2-12a^3b^3$, 2d Dividend.

$a^6-6a^5b+15a^4b^2-2a^3b^3+15a^2b^4-6ab^5+b^6$

0

2. Extract the fifth root of $32a^5-80a^4x+80a^3x^2-40a^2x^3+10ax^4-x^5$.

Ans. $2a-x$.

NOTE.—Since a^4 is $(a^2)^2$, it is evident, that in order to extract the fourth root, we might extract the square root twice. $\sqrt{a^4}=a^2$, and $\sqrt{a^2}=a$, or, which is the same thing, $\sqrt{\sqrt{a^4}}=a$. We have also, $\sqrt[3]{a^6}=a$, and $\sqrt[3]{\sqrt{a^6}}=a$. Hence, to extract the sixth root, we may first extract the cube root, and afterwards the square root; and in general, where the index of a root is composed of factors, we can extract it by extracting, successively, the roots which have these factors for indices.

The fourth root may be obtained by extracting the square root twice; the eighth root may be obtained by extracting the square root four times; the sixth root, by extracting the square and cube roots, successively; the ninth, by extracting the cube root twice, &c.

EXAMPLES.

1. Extract the fourth root of $16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4$.

Square root.

$$\begin{array}{r|l}
 & 16a^4+96a^3b+216a^2b^2+216ab^3+81b^4 \,\|\, 4a^2+12ab+9b^2 \\
 & 16a^4 \\ \hline
8a^2+12ab & 96a^3b+216a^2b^2 \\
 & 96a^3b+144a^2b^2 \\ \hline
8a^2+24ab+9b^2 & 72a^2b^2+216ab^3+81b^4 \\
 & 72a^2b^2+216ab^3+81b^4 \\ \hline
\end{array}$$

4th root.

$$\begin{array}{r|l}
 & 4a^2 + 12ab + 9b^2 \,\|\, 2a + 3b \\
 & 4a^2 \\ \hline
4a^2 + 3b & +12ab + 9b^2 \\
 & 12ab + 9b^2 \\ \hline
\end{array}$$

2. Extract the 6th root of $x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$. Ans. $x - 2$.

OF THE RULES FOR EXTRACTING THE ROOTS OF NUMBERS.

(50.) The rules which we have given for extracting the roots of algebraic quantities, are the same, in effect, as those usually given for extracting the roots of numbers. The only difference arises from the circumstance, that in numbers the different orders of units, are, by means of the decimal notation, confounded into one sum, while the quantities expressed by letters in algebra are kept distinct.

Let a represent the tens in any number, and b the units, we shall have, for the square of $a + b$, $a^2 + 2ab + b$; that is, the square of the tens, twice the product of the tens by the units, plus the square of the units. Let $a = 8$, and $b =$

then, since a represents the tens, and b the units, $a+b$ becomes $80+5=85$; and squaring these quantities, we have

$$(80+5)^2=(85)^2,$$

or, $$(80)^2+2(80)\times 5+(5)^2=7225;$$

or, $$6400+800+25=7225.$$

Now, the square root of $6400+800+25$ may be extracted according to the rules for algebraic quantities. Applying the rule we have,

$$\begin{array}{r|l} & 6400+800+25\,(80+5, \text{ or } 85, \text{ the root.} \\ & 6400 \\ \hline 160+5 & 800+25 \\ & 800+25 \\ \hline & 0 \quad 0 \end{array}$$

But, by observing that 7225 must also contain the square of the tens, + twice the product of the units by the tens, + the square of the units, expressed by means of the decimal notation; we shall be able to extract the root, after finding what part of this number expresses these several quantities. Now, if we divide this number into periods of two figures each, beginning at the right, it will be evident that the first period can form no part of the square of the tens, for $(10)^2=100$, $(20)^2=400$; and, in general, the square of the tens can have no significant figure less than hundreds; hence the square of the tens must be found in 72, the second period.

$$\begin{array}{r|l} & 72{\cdot}25 \parallel 85 \\ & 64 \\ \hline 165 & 825 \\ & 825 \\ \hline & 0 \quad 0 \end{array}$$

The greatest *square* in 72 is 64, the root of which is 8. Subtracting the square of 8 from 72, we have a remainder of 8, which, from the place it occupies, is 8 hundreds. Bringing down the next period, we have, for a remainder, after the the square of the tens is taken out, 825. This must contain the double product of the tens by the units, plus the square of the units. If we could separate the square of the units from the double product in this number, we should, evidently, find the units by dividing this double product by twice the tens;

but as this cannot be done, we cannot exactly determine the units, except by trial. In the example above, I take 16, the double of the tens for a divisor, and see how many times it is contained in 82, the tens of the remainder, since the product of tens by units must be tens. I find a quotient of 5, which I place on the root, and also in the units place in the divisor. Multiplying the divisor, so increased, the product will be the double product of the tens by the units, + the square of the units; for, 5 × 165 is the same as 5 × 160 + 5 × 5, or 5 times 16 tens, + 5 times 5 units. If 5 × 165 had been greater than the dividend, it would have been necessary to have put a figure less than 5 in the root.

If the 0s had been retained, instead of giving each figure its value depending on its place in the decimal notation, the operation would have been thus expressed:—

7225 || 80 + 5, the root = 85.
6400, square of the tens.

160 + 5 | 825

800, double product of the tens and units.
25, square of the units.

825

0 0

When the number contains more places of figures, the same principles will apply, for $(100)^2 = 1.00.00$ and $(1000)^2 = 1.00.00.00$; and hence the square of the hundreds can have no significant figures in the first two periods, nor the square of the thousands in the first three periods.

For another example, let the square root of 399424 be extracted.

39.94.24 || 63.2
36

123 | 394
369

1262 | 25.24
25.24

In this example, the root will contain three places of
since the number itself has more than four places

and therefore will contain the square of hundreds. But in order to extract the root, we may consider it as composed of tens and units only; as for example, 342 may either be considered as 3 hundreds, 4 tens, and 2 units, or 34 tens, and 2 units. Having, therefore, separated the two right hand figures, as forming no part of the square of the tens, I proceed to extract the root of 39.94. For this purpose, 3994 may be considered as so many units, and the root extracted as in the preceding example. I find the root of this number to be 63, with a remainder of 25. Bringing down to this, the right hand period, I find, for the whole remainder, 2524. This must contain the double product of the units by the tens, + the square of the units: 63 is the tens of the root; and the double of this, 126, will be the divisor to be used in finding the units. If the number contained seven or eight places of figures, there would then be four periods, and the root of the first three being extracted, it might be considered as so many tens, and the remainder of the root units, as in the preceding examples.

For the rule for extracting the square root of numbers, see Arithmetic, art. (56), page 67.

EXAMPLES.

1. $\sqrt{859329} = 927.$
2. $\sqrt{390625} = 625.$
3. $\sqrt{998001} = 999.$
4. $\sqrt{52441} = 229.$
5. $\sqrt{\frac{529}{841}} = \frac{23}{29}.$

EXTRACTION OF THE CUBE ROOT OF NUMBERS.

(51.) Let a represent the tens of any number, and b the units; then, $(a+b)$ will be that number, and

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Hence it will appear, that the cube root of any number is composed of the cube of its tens, + three times the square of the tens multiplied by the units, + three times the tens multiplied by the square of the units, + the cube of the units. If we take any number, 24 for example, we can put it under the form 20 + 4, and raising it to the third power, we shall have

$$(20+4)^3 = (20)^3 + 3(20)^2 \times 4 + 3(20 \times (4)^2 + 4^3$$
$$= 8000 + 4800 + 960 + 64 = 13824.$$

$$\begin{array}{r|l} & 13.824 \;\|\; 24 \\ & 8 \\ \hline 1200 & 5824 \\ 240 & \\ 16 & \\ \hline 1456 & 5824 \end{array}$$

Now, in order to extract the cube root of 13.824, it will be necessary to discover what part of this number contains the cube of the tens, and since $(10)^3 = 1000$, it is evident that the three right hand figures make no part of the cube of the tens. We must, therefore, find the cube of the tens in 13. Now the greatest cube in 13 is 8, which is the cube of 2. Putting 2 in the root, and subtracting its cube, we have 5824 remaining for the remaining terms; namely, three times the square of the tens multiplied by the units, three times the tens multiplied by the square of the units, and the cube of the units; but since the first of these terms is much the largest, we shall be able to obtain the units nearly, by dividing 5824 by 1200, which is three times the square of the tens. Having thus found the unit figure, which is 4, we can verify the result by forming with it and the tens, the three products represented by $3a^2b+3ab^2+b^3$; but this is the same as $(3a^2+3ab+b^2)b$. If, therefore, we add to 1200, the divisor $= (3a^2)$, three times 20×4, or $240 = (3ab)$, and the square of 4, or $16 = (b^2)$, and multiply the sum by $4 = (b)$, the product will be 5824. If the product had been greater than the dividend, it would have been necessary to put a less figure in the units place. The ciphers in 1200 and 240 might have been omitted, but in that case, the figures must be written in the places corresponding to their values; thus,

$$\begin{array}{r} 12 \\ 24 \\ 16 \\ \hline 1456 \end{array}$$

The reason for these values is obvious; for the tens can have no figure less than hundreds, and the product of the tens by the units, none less than tens.

If the number contains more places of figures, the same reasoning will apply as in the square root. The number must be divided into periods of three figures each; for we have $(10)^3 = 1000$, $(100)^3 = 1.000.000$, $(1000)^3 = 1.000.000.000$, from which it will appear, that the cube of tens can have no significant figure in the first period, the cube of hundreds none in the first two, and the cube of thousands none in the first three. The number of periods will, therefore, denote the number of figures in the root.

Let it be required to extract the cube root of 199176704.

		199.176.704(584
		125
Div. $(5)^2 \times 3 = 25 \times 3 =$	75.00	74.176 1st dividend.
$8 \times 5 \times 3 =$	120.0	
$(8)^2 =$	64	
	8764	70.112
Complete divisor.		
2d div. $(58)^2 \times 3 = 3364 \times 3 =$	10092.00	4064704 2d dividend.
$58 \times 4 \times 3 =$	696.0	
$(4)^2 =$	16	
2d complete divisor.	1016176	4064704

In this example, I first seek the root of the two left hand periods, which I find to be 58. I then call this 58 tens, and proceed to find the last figure of the root by taking three times the square of 58 tens, for a divisor, and bringing down the last period to the remainder, for a dividend. The divisor is completed in the same manner as in the last example. If the given number contained four periods, the same method would be followed, always considering the figures of the root, already found, so many tens, and the figure sought, so many units.

For the rule for extracting the cube root of numbers, see Arith. art. (57), p. 70.

EXAMPLES.

1. $\sqrt[3]{7645373} = 197$.
2. $\sqrt[3]{52734375} = 375$.
3. $\sqrt[3]{334255384} = 694$.

GENERAL METHOD OF EXTRACTING THE ROOTS OF NUMBERS OF ANY POWERS WHATEVER.

(52.) The roots of any powers whatever may be extracted by a method analogous to that given (48) for extracting the roots of algebraic quantities.

The first two terms of the developement of any power of a binomial, as $(a+b)^n$ is always $a^n + na^{n-1}b$ (43). Hence it will appear, that if we make $a =$ the tens of any number, and $b =$ the units, we can approximate to the units by raising the tens to a power one less, or a^{n-1}, and multiplying this power by n, the index of the root to be extracted, for a divisor. To find in what part of the number we are to look for the power of the tens, we are to point off the number in periods of n figures each; for, in raising 10 to any power, we add a cipher for every multiplication; thus, $(10)^2 = 100$, $(10)^3 = 1000$, $(10)^4 = 10000$; and since in raising 10 to the nth power, it is to be multiplied by itself $n - 1$ times, we shall have for the nth power of 10, 1 and n ciphers. Hence, the nth power of the tens of any number, can have no significant figure in the first n figures, on the right.

Let it now be required to extract the 4th root of 331776.

$$\begin{array}{rl}
 & 33.1776(24 \\
 & 16 \\
 & \overline{} \\
(2)^3 = 8 \times 4 = \overset{\text{div.}}{32}) & 17.9376 \\
 & \overline{} \\
(24)^4 = & 33.1776
\end{array}$$

In this example, $n = 4$; I therefore point off four figures from the right, as containing no part of the fourth power of the tens. The greatest fourth power in the left hand period is 16, the fourth power of 2. I therefore conclude that 2 is the tens of the root: to find the units, I raise 2 to the third power, and multiply it by 4, the index of the root, for a divisor. It then corresponds to na^{n-1} of the developement of

$(a+b)^n$. For a dividend, I bring down to the remainder the last period. I have now, for a divisor, 32, and for a dividend, 179.376; and since 32 is four times the cube of the *tens*, it can occupy no lower place than that of thousands. I therefore reject the last three places of figures in the dividend, and see how many times the divisor is contained in 179. I find the quotient to be 5. But the dividend being what remains after the fourth power of the tens is taken out, corresponds to $4a^3b+6a^2b^2+4ab^3+b^4$, the first term of which, divided by $4a^3$, which in this case is 32, will give, for a quotient, b the units of the root. It is evident, therefore, that the whole of this dividend divided by 32, will give a quotient somewhat too large, and on this account I take 4, instead of 5, for the units of the root. Having raised 24 to the fourth power, I find it exactly equal to the given number, which shows that it is the root required.

The greater the index of the root to be extracted, the more will the quotient exceed the units of the root, for the higher the power of a binomial, the greater will be the number of terms remaining after the first term is taken away. If there are more periods than one, the root will consist of more than two places of figures; but the same principles will apply; for having obtained two figures of the root, we may consider them as expressing so many tens, and proceed as before.

RULE.

1. *Divide the number into periods of as many figures each, as there are units in the index of the root.*

2. *Extract the root of the left hand period, and subtract its power from that period, and to the remainder bring down the first figure of the next period for a dividend.*

3. *Involve the root, already found, to a power one less than the index of the required root, and multiply this power by the index of the root, for a divisor.*

4. *See how many times this divisor is contained in the dividend, and place the result in the root.*

5. *Involve the whole root to the given power, and subtract it from the two left hand periods, and to the remainder bring down the first figure of the next period for a new dividend, and form a divisor from the whole root already found, in the same manner as before; proceed in like manner, till all the periods are brought down, always observing to subtract the root when involved to the given*

power, from as many of the left hand periods of the given number, as there are figures in the root.

EXAMPLES.

1. Extract the cube root of 148877.

$$148\dot{8}77(53$$
$$125$$
Div. $3(5)^2 = 3 \times 25 = 75)238$
$(53)^3 = 1488,77$

2. Extract the cube root of 176558481.

$$176\dot{5}58\dot{4}81(561$$
$$125$$
Div. $3(5)^2 = 75)515$ dividend.
$(56)^3 = 175616$
2d Div. $3(56)^2 = 9408)9424$ 2d dividend.
$(561)^3 = 176558481$

3. Extract the fifth root of 33554432.

$$335,54432(32$$
$$243$$
Div. $5 \times (3)^4 = 405)925$ dividend.
$(32)^5 = 33554432$

NOTE.—If the index of the root to be extracted, be such a number as can be decomposed into factors, we can obtain the root by successively extracting the roots which have these factors for indices, upon the principles laid down in Art. (49).

EXAMPLES.

1. $\sqrt[6]{244140625} = \sqrt{\sqrt[3]{244140625}} = \sqrt[3]{15625} = 25.$

2. $\sqrt[8]{429981696} = \sqrt{\sqrt[4]{429981696}} = \sqrt{\sqrt{20736}} = 12.$

3. $\sqrt[9]{19683} = 3.$

4. $\sqrt[6]{887503681} = 31$

5. $\sqrt[4]{985036} = 31{,}527.$

6. $\sqrt[8]{390625} = 5.$

EXTRACTION OF ROOTS BY APPROXIMATION.

(53.) There are but few numbers, comparatively, that will admit of exact roots; these are called perfect powers.

1, 4, 9, 16, 25, 36, 49, &c.
1, 8, 27, 64, 125, 216, 343, &c.

are perfect squares and cubes; but all the intermediate numbers will admit of no exact root, it being impossible to find any number, which, multiplied by itself once or twice, will produce any one of those numbers.

Indeed, it can be shown, that the root of a number which is not a perfect power, can be neither a whole number nor a fraction. In the first place, the root cannot be a whole number, for then the number itself would be a perfect power of that root, which is contrary to the supposition.

To show that the root of an imperfect power cannot be a fraction, it is necessary to prove that the powers of any irreducible fraction, as $\frac{c}{b}$, are irreducible. To this purpose, we remark, that the terms of such a fraction being prime to each other, they can have no factor in common; and hence, their prime factors must be different. Let the prime factors of c, be, $d\,e \ldots . f$, and the prime factors of b, $h\,k \ldots . l$; then,

$$\frac{c}{b} = \frac{d \times e \times \ldots . f}{h \times k \times \ldots . l}$$

Raising both members of this equation to the *n*th power, we have

$$\frac{c^n}{b^n} = \frac{d^n \times e^n \times \ldots . f^n}{h^n \times k^n \times \ldots . l^n}$$

In raising the second member of this equation to the *n*th power, we have introduced no new factor; we have only repeated each of the prime factors which belong to the terms of the fraction, *n* times. The numerator and denominator have, therefore, no prime factor, in common; therefore $\frac{c^n}{b^n}$ is an irreducible fraction.

This being premised, suppose it possible that the *n*th root of a number, *a*, for example, be a fraction, and that this fraction, reduced to its lowest terms, is $\frac{c}{b}$; we have $\sqrt[n]{a} = \frac{c}{b}$. Raising both members of this equation to the *n*th power, we have

$$a = \frac{c^n}{b^n};$$

but $\frac{c}{b}$ being an irreducible fraction, $\frac{c^n}{b^n}$ will also be irreducible, as shown above; therefore *a* cannot be equal to $\frac{c^n}{b^n}$, for, in that case, we should have a whole number equal to an irreducible fraction, which is impossible. Therefore $\sqrt[n]{a}$ cannot be equal to $\frac{c}{b}$, or any root of a whole number be equal to a fraction.

But though we cannot obtain the exact root of a number which is not a perfect power, we can approximate as near to it as we please. For example; let it be required to extract the square root of 5 to within $\frac{1}{10}$. We may write 5 in the form of a fraction; thus $\frac{5}{1}$, and by multiplying both numerator and denominator by the square of 10 = 100, we shall have $\frac{500}{100}$; now, by extracting the root of both numerator and denominator, we have 10 for the denominator, and the numerator between 22 and 23. $\frac{22}{10}$ is too small, and $\frac{23}{10}$ too large, so that the root is determined to within one-tenth. If we would determine the cube root of any number, 14 for example, to within $\frac{1}{10}$, we may put it in the fractional form, $\frac{14}{1}$, and multiply both terms by 1000, the cube of 10, and we shall have $\frac{14000}{1000}$; the cube root of which, is between $\frac{24}{10}$ and $\frac{25}{10}$, or within $\frac{1}{10}$ of the exact root. The same reasoning will apply to any root whatever, and any number besides 10 may be used for the denominator of the root; for if we would extract the *n*th root of a to within $\frac{1}{m\text{th}}$, we have only to write a in the fractional form, $\frac{a}{1}$, and multiply both terms by the *n*th power of m. We shall then have $\frac{am^n}{m^n}$. Now the *n*th root of the denominator is m, and calling r the nearest root of the numerator, the root will be between $\frac{r}{m}$ and $\frac{r+1}{m}$; that is, within $\frac{1}{m\text{th}}$ of the exact root.

(54.) We will now apply this principle to the extraction of roots by approximation in decimals. We found the square root of 5 to be $\frac{22}{10}$, or 2,2. Had it been required to approximate to within $\frac{1}{100}$ of the exact root, we should have multiplied both terms of the fraction by $(100)^2$, 10000, and it would have become $\frac{50000}{10000}$, the square root of which is $\frac{223}{100}$, or 2,23, which is the root to within $\frac{1}{100}$. If it were required to extract the cube root of any number to within $\frac{1}{10}$, $\frac{1}{100}$, or $\frac{1}{1000}$, it is evident that it might be done by multiplying that number by the cube of 10, 100, or 1000, and extracting the root. The root obtained would be 10ths, 100ths, or 1000ths, which might either be written in the form of a vulgar fraction, or, by pointing off the requisite number of figures, in the form of a decimal fraction. Thus, the cube root of

$$31 = \sqrt[3]{\frac{31 \times (100)^3}{(100)^3}} = \sqrt[3]{\frac{31000000}{1000000}} = \frac{314}{100} = 3{,}14.$$

In these examples, the denominators might have been omitted, and the given number multiplied by the square, or cube, of 10, 100, 1000, &c. The number of 0's added to the given number, will always show the number of decimal places in the root. For multiplying by $(10)^2 = 100$, is the same as adding two 0's to the given number. Multiplying by $(100)^2$, which is equal to $(10)^2 \times (10)^2$, is the same as adding four 0's; multiplying by $(1000)^2 = (10)^2 \times (10)^2 \times (10)^2$, is adding six 0's; and hence, in extracting the square root, it is evident, that for every two 0's added, there will be one place of decimals in the root.

In extracting the cube root we have, $(10)^3 = 1000$, $(100)^3 = (10)^3 \times (10)^3$; $(1000)^3 = (10)^3 \times (10)^3 \times (10)^3$, &c.; and hence, for every three 0's added to the given number, we shall have one place of decimals. And, in general, since the *n*th power of ten has no 0's we shall always have, in extracting the *n*th root, one place of decimals for every *n* 0's added to the given number, for $(10)^n \times 10^n = (100)^n$, $10^n \times 10^n \times 10^n = (1000)^n$, &c.

For extracting roots by approximation in decimals, we have, therefore, the following general

RULE.

For each place of decimals required in the root, add as many 0's as there are units in the index of the root to be extracted.

EXAMPLES.

1. Extract the square root of 12, to three places of decimals.

$$\begin{array}{r|l}
 & 12,00,00,00(3,464 \\
 & 9 \\
\hline
64 & 300 \\
 & 256 \\
\hline
686 & 4400 \\
 & 4116 \\
\hline
6924 & 28400 \\
 & 27696 \\
\hline
 & 704
\end{array}$$

2. Extract the cube root of 15 to two places of decimals.

$$\begin{array}{r|l}
 & 15.000.000 \,\|\, \overset{\text{Root.}}{2.46} \\
 & 8 \\ \cline{2-2}
1200 & 7000 \\
240 & \\
16 & \\ \hline
1456 & 5824 \\ \hline
172800 & 1176000 \\
43200 & \\
36 & \\ \hline
177156 & 1062936 \\ \cline{2-2}
 & 113064
\end{array}$$

3. Extract the 4th root of 25 to one place of decimals.

$$\begin{array}{r|l}
 & 25,0000 \,\|\, \overset{\text{Root.}}{2,2} \\
 & 16 \\ \cline{2-2}
(2)^3 \times 4 = 32 & 90,000 \\ \cline{2-2}
(22)^4 = & 234256 \\ \cline{2-2}
 & 15744
\end{array}$$

4. Extract the cube root of 2, to six places of decimals.
Ans. 1,259921.

5. Extract the cube root of 5, to three places of decimals.
Ans. 1,709.

6. Extract the fourth root of 38, to two places of decimals.
Ans. 2,47.

To extract the root of a whole number and decimal fraction, as 15,173, it is always necessary to affix so many 0's to the right, as will make out one or more complete periods of the root, counting from the decimal point; for if we have the number 15,173, as above, and which is the same as $\frac{15173}{1000}$; it will be perceived that the denominator will not admit of an exact root. But if we multiply both terms of the fraction by 10, we shall have $\frac{15,1730}{10000}$, and the denominator being now a perfect square, its root can be extracted, and will give 100 for the denominator of the root; or, which is the same thing, two places of decimals in the root of the given number. The same reasoning will apply to other roots. Let it be required to extract the cube root of 1,7231, or $\frac{17231}{10000}$. To make the denominator a perfect cube, it must be multiplied by 100, and the fraction then becomes $\frac{1723100}{1000000}$, or 1,723,100, the root of which can be extracted by the preceding rules, and gives two places of decimals in the root. Hence, for the extraction of the roots of decimals, we have the following

RULE.

Divide the decimal part into periods, corresponding to the index of the root, beginning at the decimal point; and if there is not the requisite number of figures, fill out the last period by affixing 0's, and extract the root as in whole numbers.

EXAMPLES.

1. Extract the square root of ,0034672.

```
                 Root.
,00.34.67.20 || ,0588
    25
   ------
108 | 967
    | 864
    -------
1168 | 10320
     |  9344
      ------
         976
```

2. **Extract the cube root of 13,0125.**

$$\begin{array}{r|l}
 & \text{Root.} \\
 & 13{,}012.500 \;\|\; 2{,}35 \\
 & 8 \\ \hline
1200 & 5012 \\
180 & \\
9 & \\ \cline{1-1}
1389 & 4167 \\ \hline
158700 & 845500 \\
3450 & \\
25 & \\ \cline{1-1}
162175 & 810875 \\ \cline{2-2}
 & 34625
\end{array}$$

3. **Extract the fourth root of 7,82657.**

$$\begin{array}{r|l}
 & \dot{7}{,}8\dot{2}.6\dot{5}.7\dot{0}.00 \;\|\; 2{,}7976, \text{ the square root.} \\
 & 4 \\ \cline{2-2}
47 & 382 \\
 & 329 \\ \cline{2-2}
549 & 5365 \\
 & 4941 \\ \cline{2-2}
5587 & 42470 \\
 & 39109 \\ \cline{2-2}
55946 & 336100 \\
 & 335676 \\ \cline{2-2}
 & 424
\end{array}$$

$$\begin{array}{r|l}
 & 2{,}79.76 \;\|\; 1{,}67 \text{ 4th root.} \\
 & 1 \\ \cline{2-2}
26 & 179 \\
 & 156 \\ \cline{2-2}
327 & 2376 \\
 & 2289 \\ \cline{2-2}
 & 65
\end{array}$$

4. **Extract the square root of 2,34572.**

Ans. 1,5315.

CALCULUS OF RADICALS.

(54.) In the extraction of the roots of algebraic quantities, it is often the case, that the quantity will not admit of an ex-

act root; such quantities are called ***radicals***, or ***irrational*** quantities, and are written with the sign of the root, called also the radical sign. $\sqrt{2ab}$, $2\sqrt[3]{a^2}$, $a\sqrt[4]{6a^3}$ are radicals.

The figure written before, and a little above the radical sign, shows the degree of the root to be extracted, and is called the *index* of the radical. In writing the sign for the square root, the index is generally omitted.

Two radicals are *similar* when they have the same indices, and also the same quantities under the radical sign: $\sqrt{5}$ and $7\sqrt{5}$, $\sqrt[3]{ac}$ and $2\sqrt[3]{ac}$, are similar radicals.

The coefficient of a radical shows how many times the radical is taken in the algebraic expression of which it forms a part, and is written immediately before the sign: thus, $2b$, $(3a+b)$, and 21 are respectively, the coefficients of the radicals, $2b\sqrt{3a}$, $(3a+b)\sqrt[3]{7a^2}$, and $21\sqrt[4]{5}$.

A radical is said to be reduced to its simplest form, when the quantity under the radical sign contains no factor, the root of which can be extracted.

To reduce radicals to their simplest form.

RULE.

(55.) *Decompose the quantity under the radical sign into two factors, one of which shall be a perfect power of the root to be extracted; then extract the root of this factor, and multiply the coefficient of the radical by the root. The other factor must remain under the radical sign.*

Let it be required to reduce $\sqrt{48a^3b}$ to its simplest form.

$$\sqrt{48a^3b} = \sqrt{16a^2 \times 3ab} = 4a\sqrt{3ab}.$$

The quantity under the radical sign can be decomposed into the factors $16a^2$ and $3ab$. The first of these is a perfect square, and its root is $4a$. Leaving the factor $3ab$ under the sign, and multiplying the coefficient of the radical, which, in this example, is 1, by $4a$, we have $4a\sqrt{3ab}$ for the radical reduced to its simplest form.

This rule is founded upon the principle, that *the root of any quantity is equal to the product of the root of its tors.*

To prove this, take the quantity $\sqrt[n]{abc}$, which is the *n*th root of *abc*, and we say that it is equal to the *n*th root of *a*, multiplied by the *n*th root of *b*, multiplied by the *n*th root of *c*; and that we shall have,

$$\sqrt[n]{abc} = \sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c};$$

for if we raise both quantities to their *n*th powers, we shall have

$$\sqrt[n]{abc})^n = \sqrt[n]{a^n} \times \sqrt[n]{b^n} \times \sqrt[n]{c^n};$$

or, $\quad abc = a \times b \times c = abc.$

The *n*th powers of these quantities being, therefore, equal, it is evident that the quantities themselves are equal. Now, in taking a quantity from under the radical sign, we extract the root of one of the factors of the radical; and since we multiply the coefficient of the radical by this root, the expression is still the product of the roots of the factors. Thus, $\sqrt{a^2b}$ is the product of the square roots of a^2 and b; and $a\sqrt{b}$ is evidently the same; the only difference is in the form of the expression. In the first, the square root of a^2 is indicated by the sign, and in the last, it has been actually extracted.

EXAMPLES.

1. Reduce $\sqrt[3]{96a^5b^4c}$ to its simplest form.

$$\sqrt[3]{96a^5b^4c} = \sqrt[3]{8a^3b^3 \times 12a^2bc} = 2ab\sqrt[3]{12a^2bc}.$$

2. Reduce $\sqrt[4]{243a^2b^5}$ to its simplest form.

$$\sqrt[4]{243a^2b^5} = \sqrt[4]{81b^4 \times 3a^2b} = 3b\sqrt[4]{3a^2b}.$$

3. Reduce $\sqrt{3a^2b + 6ab^2 + 3b^3}$ to its simplest form.

$$\sqrt{3a^2b+6ab^2+3b^3} = \sqrt{(a^2+2ab+b^2)3b} = (a+b)\sqrt{3b}.$$

4. Reduce $\sqrt[3]{8a^4b^2 + 24a^3b}$ to its simplest form.

$$\sqrt[3]{8a^4b^2+24a^3b} = \sqrt[3]{8a^3(ab^2+3b)} = 2a\sqrt[3]{ab^2+3b}.$$

5. Reduce $\sqrt{54ab^2c^3}$ to its simplest form.

Ans. $3bc\sqrt{6ac}.$

6. Reduce $\sqrt[4]{32a^6b^2c^4}$ to its simplest form.

Ans. $2ac\sqrt{2a^2b^2}$.

7. Reduce $4\sqrt[3]{16a^4c^3b}$ to its simplest form.

Ans. $8ac\sqrt[3]{2ab}$.

8. Reduce $2\sqrt{\frac{3}{16}}$ to its simplest form.

$$2\sqrt{\tfrac{3}{16}}=2\sqrt{\tfrac{1}{16}\times 3}=2\times\tfrac{1}{4}\sqrt{3}=\tfrac{1}{2}\sqrt{3},\text{ the ans.}$$

9. Reduce $\sqrt[3]{\frac{11ab^3}{16c^3}}$ to its simplest form.

Ans. $\frac{b}{2c}\sqrt[3]{\frac{11a}{2}}$.

NOTE.—If the quantity under the radical sign be a fraction, it can always be changed to a whole number, by multiplying both terms of the fraction by such a quantity as shall make the denominator a perfect power; thus,

$$\sqrt{\tfrac{1}{3}}=\sqrt{\tfrac{3}{9}}=\sqrt{\tfrac{1}{9}\times 3}=\tfrac{1}{3}\sqrt{3},$$

and $$\sqrt[n]{\frac{a}{b}}=\sqrt[n]{\frac{ab^{n-1}}{b^n}}=\sqrt[n]{\frac{1}{b^n}\times ab^{n-1}}=\frac{1}{b}\sqrt[n]{ab^{n-1}}.$$

EXAMPLES.

1. Reduce $\sqrt[3]{\frac{1}{4}}$ to its simplest form.

Ans. $\sqrt[3]{\frac{1}{4}}=\sqrt[3]{\frac{16}{64}}=\sqrt[3]{\frac{8}{64}\times 2}=\frac{1}{2}\sqrt[3]{2}$.

2. Reduce $\sqrt{\frac{1}{3}a}$ to its simplest form.

Ans. $\frac{1}{3}\sqrt{3a}$.

3. Reduce $\sqrt{\frac{27}{50}}$ to its simplest form.

Ans. $\sqrt{\frac{27}{50}}=\sqrt{\frac{54}{100}}=\sqrt{\frac{9}{100}\times 6}=\frac{3}{10}\sqrt{6}$.

4. Reduce $4\sqrt{\frac{3}{11}}$ to its simplest form.

Ans. $\frac{4}{11}\sqrt{33}$.

5. Reduce $a\sqrt[3]{\frac{a^2}{b^2}}$ to its simplest form.

Ans. $\frac{a}{b}\sqrt[3]{a^2b}$.

6. Reduce $(a+b)\sqrt{\frac{2a}{a-b}}$ to its simplest form.

Ans. $(a^2-b^2)\sqrt{2a^2-2ab}$.

7. Reduce $4\sqrt{\frac{27}{12}}$ to its simplest form.

Ans. 6.

8. Reduce $3\sqrt[3]{\frac{2}{7}}$ to its simplest form.

Ans. $\frac{3}{7}\sqrt[3]{98}$.

(56.) If we have a radical of the form $4\sqrt{15}$, and wish to find the root as nearly as possible in whole numbers, it is necessary to reverse the process described in the preceding rule, and place the coefficients under the radical sign. This is done by squaring the coefficient, and multiplying the radical by its square: thus, $4\sqrt{15}=\sqrt{16\times 15}=\sqrt{240}$. Now the entire part of the square root of 240 is 15; but if we had taken the square root of 15, the number under the radical sign, we should have found 3 for the entire part of the root, which, multiplied by 4, would give only 12 for the value of the radical in whole numbers. So also, $6\sqrt{20}=6\times 4=24$: but by putting the coefficient under the sign, we have,

$$6\sqrt{20}=\sqrt{36\times 20}=\sqrt{720}=26.$$

The same principle may be applied to any radical whatever; and hence, for putting the coefficient of a radical under the sign, we have the following

RULE.

Raise the coefficient to a power corresponding to the index of the radical, and multiply the radical by this power.

EXAMPLES.

1. $6\sqrt[3]{13}=\sqrt[3]{6^3\times 13}=\sqrt[3]{2808}$.

2. $3a\sqrt[4]{ab}=\sqrt[4]{81a^4\times ab}=\sqrt[4]{81a^5b}$.

$$3.\ (a+b)\sqrt{ab}=\sqrt{(a+b)^2\times ab}=\sqrt{a^3b+2a^2b^2+ab^3}.$$

$$4.\ \tfrac{1}{3}\sqrt{3a}=\sqrt{\frac{a}{3}}\,.$$

$$5.\ \tfrac{1}{4}\sqrt[3]{4}=\sqrt[3]{\tfrac{1}{16}}.$$

To reduce radicals of different indices, to equivalent radicals, having a common index.

RULE.

(57.) *Find the least common multiple of all the indices; this will be the common index. Divide this common index by each particular index, and raise the quantity under the radical sign, to the power denoted by the quotient.*

Let it be required to reduce $\sqrt[2]{ac}$, $\sqrt[3]{3b}$, and $\sqrt[4]{a^2b}$, to equivalent radicals having a common index. The least common multiple of the indices is 12; and if we divide this by each particular index, we obtain, for the quotients, respectively, 6, 4, and 3. The first radical is therefore to be raised to the sixth power, the second to the fourth, and the third to the third, and they become, $\sqrt[12]{a^6c^6}$, $\sqrt[12]{81b^4}$, and $\sqrt[12]{a^6b^3}$.

This rule depends upon the principle, that the value of a radical is not altered, when the index of the radical, and the exponent of the quantity under the sign, are both multiplied by the same number. To prove this, we have only to show that the $\sqrt[mn]{a^n}$ is the same as the $\sqrt[m]{a}$. We have seen (45) that the root of a quantity may be extracted by dividing the exponent of the quantity by the index of the root. According to this notation, the *mn*th root a^n, is written $a^{\frac{n}{mn}}$, which is the same as $a^{\frac{1}{m}}$, which again is equivalent to $\sqrt[m]{a}$; whence we conclude, that $\sqrt[mn]{a^n}=\sqrt[m]{a}$.

EXAMPLES.

1. Reduce $\sqrt{\frac{a}{2}}$ and $\sqrt[3]{\frac{a}{2}}$ to a common index.

Ans. $\sqrt[6]{\frac{a^3}{8}}$ and $\sqrt[6]{\frac{a^2}{4}}$.

2. Reduce $\sqrt{\frac{1}{2}}$ and $\sqrt[4]{\frac{1}{4}}$ to a common index.

Ans. $\sqrt[4]{\frac{1}{4}}$ and $\sqrt[4]{\frac{1}{4}}$.

3. Reduce $\sqrt[3]{a^2bc}$ and $\sqrt[5]{\frac{2c^2}{a}}$ to a common index.

Ans. $\sqrt[15]{a^{10}b^5c^5}$ and $\sqrt[15]{\frac{8c^6}{a^3}}$.

4. Reduce $\sqrt{a}$, $\sqrt[3]{b}$, $\sqrt[4]{c}$, and $\sqrt[6]{d}$ to a common index.

Ans. $\sqrt[12]{a^6}$, $\sqrt[12]{b^4}$, $\sqrt[12]{c^3}$, and $\sqrt[12]{d^2}$.

5. Reduce $\sqrt[3]{\frac{3}{4}}$, $\sqrt{\frac{1}{3}}$, and $\sqrt[4]{2}$ to a common index.

Ans. $\frac{1}{2}\sqrt[12]{1296}$, $\frac{1}{3}\sqrt[12]{6561}$, and $\sqrt[12]{8}$.

6. Reduce $\sqrt[3]{3\frac{1}{2}}$ and $\sqrt[4]{5\frac{1}{3}}$ to a common index.

Ans. $\sqrt[12]{150\frac{1}{16}}$ and $\sqrt[12]{151\frac{19}{27}}$.

ADDITION AND SUBTRACTION OF RADICALS.

RULE.

(58.) *If the radicals are* similar, *add or subtract their coefficients; but if they are not similar, the addition or subtraction can only be indicated.*

EXAMPLES.

1. Find the sum of $a\sqrt{2ac}$, $2\sqrt{2ac}$, and $3a\sqrt{2ac}$.

Ans. $(4a-2)\sqrt{2ac}$.

2. Find the sum of $3a\sqrt[3]{2bc}$, $7\sqrt[3]{11a^2}$, and $12b\sqrt[3]{19c^2}$.

Ans. $3a\sqrt[3]{2bc}+7\sqrt[3]{11a^2}+12b\sqrt[3]{19c^2}$.

3. Find the difference between $2b\sqrt[3]{a^2b}$ and $a\sqrt[3]{a^2b}$.

Ans. $(2b-a)\sqrt[3]{a^2b}$.

4. Add together $3\sqrt{ab^3c^2}$, $c\sqrt{ab^3}$, and $4b\sqrt{abc^2}$.

These radicals, in their present form, are not similar; but by reducing them to their simplest form, they become, respectively, $3bc\sqrt{ab}$, $bc\sqrt{ab}$, and $4bc\sqrt{ab}$. Their sum is, therefore, $8bc\sqrt{ab}$.

5. From $11\sqrt{96a^2bc^3}$ subtract $3a\sqrt{24bc^3}$.

Ans. $38ac\sqrt{6bc}$.

6. Find the sum of $4\sqrt{a+x}$ and $\sqrt{4a^3b^2+4a^2b^2x}$.

Ans. $(4+2ab)\sqrt{a+x}$.

7. Find the sum of $3\sqrt[3]{16a^4b^3}$ and $ab\sqrt[3]{54a}$.

Ans. $9ab\sqrt[3]{2a}$.

8. Find the sum of $3\sqrt{3}$ and $2\sqrt{\frac{1}{3}}$.

Ans. $\frac{11}{3}\sqrt{3}$.

9. From $\sqrt[3]{192}$ subtract $\sqrt[3]{24}$. Ans. $2\sqrt[3]{3}$.

10. Find the sum $\sqrt{\frac{3}{4}}$, $4\sqrt{12}$, and $3\sqrt{\frac{1}{12}}$. Ans. $9\sqrt{3}$.

11. From $3\sqrt[3]{\frac{2}{5}}$ subtract $\frac{1}{4}\sqrt[3]{400}$. Ans. $\frac{1}{10}\sqrt[3]{50}$.

MULTIPLICATION OF RADICALS.

RULE.

(59.) *Reduce the radicals to a common index, and then multiply the quantities under the sign, together, and place the product under the common radical sign. If there are coefficients, these must also be multiplied together.*

This rule depends upon the principle demonstrated in art. (56), that the *n*th root of the product of two or more quantities, is equal to the product of the *n*th roots: thus,

$$\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab};$$

for, raising both quantities to the *n*th power, we have,

$$a\times b=ab.$$

EXAMPLES.

1. Multiply $3a\sqrt{2bc}$ by $5b\sqrt{3c}$.

$$3a\sqrt{2bc}\times 5b\sqrt{3c}=15ab\sqrt{2bc\times 3c}=15ab\sqrt{6bc^2}.$$

2. Multiply $\sqrt{3ab}$ by $2\sqrt{7a^2c}$.

Reduced to a common index, these radicals become

$$\sqrt[6]{27a^3b^3},\text{ and } 2\sqrt[6]{49a^4c^2};\text{ and }\sqrt[6]{27a^3b^3}\times 2\sqrt[6]{49a^4c^2}$$
$$=2\sqrt[6]{1323a^7b^3c^2}=2a\sqrt[6]{1323ab^3c^2}.$$

3. Multiply $4\sqrt{2}$ by $\sqrt[3]{6}$. Ans. $4\sqrt[6]{288}$.

4. Multiply $5a\sqrt[3]{a+x}$ by $4b\sqrt[3]{(a+x)^2}$.

Ans. $20ab\sqrt[3]{(a+x)^3}=20ab(a+x)$.

5. Multiply $\sqrt[3]{\frac{1}{2}}$ by $\sqrt{\frac{1}{3}}$.

Ans. $\sqrt[6]{\frac{1}{108}}$ or $\frac{1}{6}\sqrt[6]{432}$.

6. Multiply $5\sqrt[4]{3a^2}$ by $2\sqrt[4]{27a^2b}$. Ans. $30a\sqrt[4]{b}$.

7. Multiply $\sqrt{\frac{3ab}{2c}}$ by $b\sqrt{\frac{ab}{c}}$. Ans. $\frac{ab^2}{2c}\sqrt{6}$.

8. Multiply $4\sqrt[3]{\frac{1}{16}}$ by $3\sqrt{8}$. Ans. $12\sqrt[6]{2}$.

9. Multiply $2\sqrt{\frac{12}{7}}$ by $\frac{1}{2}\sqrt[4]{\frac{2}{9}}$. Ans. $\sqrt[4]{\frac{288}{441}}=2\sqrt[4]{\frac{18}{441}}$.

10. Multiply $\sqrt[4]{64}$ by $2\sqrt{2}$. Ans. 8.

DIVISION OF RADICALS.

RULE.

(60.) *Reduce the radicals to a common index; then divide the quantities under the sign, and place the quotient under the common radical sign. If the radicals have coefficients, the coefficient of the dividend must be divided by that of the divisor.*

EXAMPLES.

1. Divide $a\sqrt[3]{4a^2b}$ by $b\sqrt[3]{2ab}$.

$$a\sqrt[3]{4a^2b} \div b\sqrt[3]{2ab} = \frac{a}{b}\sqrt[3]{\frac{4a^2b}{2ab}} = \frac{a}{b}\sqrt[3]{2a}.$$

2. Divide $\sqrt{4}$ by $\sqrt[3]{4}$.

These radicals, reduced to a common index, are $\sqrt[6]{64}$, and $\sqrt[6]{16}$, and $\sqrt[6]{64} \div \sqrt[6]{16} = \sqrt[6]{4}$.

3. Divide $\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{1}{3}}$. Ans. $\sqrt{\frac{3}{2}}$, or $\frac{1}{2}\sqrt{6}$.

4. Divide $9\sqrt{2b}$ by $3\sqrt{\frac{1}{b}}$. Ans. $3b\sqrt{2}$.

5. Divide $\frac{1}{2}\sqrt[3]{\frac{1}{2}}$ by $\frac{1}{3}\sqrt[3]{\frac{1}{3}}$. Ans. $\frac{3}{4}\sqrt[3]{12}$.

6. Divide $\sqrt{a^2-b^2}$ by $\sqrt[4]{a+b}$.

Ans. $\sqrt[4]{\frac{(a^2-b^2)^2}{a+b}} = \sqrt[4]{\frac{(a^2-b^2)(a^2-b^2)}{a+b}} = \sqrt[4]{(a-b)(a^2-b^2)}$.

7. Divide $\sqrt{12}$ by $\sqrt[3]{24}$. Ans. $\sqrt[6]{3}$.

8. Divide $4\sqrt[6]{32}$ by $\sqrt[3]{16}$. Ans. $2\sqrt[6]{8}$.

9. Divide $2\sqrt{8} \times 4\sqrt[4]{\frac{1}{4}}$ by $4\sqrt[3]{\frac{1}{2}} \times 4\sqrt[6]{4}$. Ans. 1.

INVOLUTION OF RADICALS.

RULE.

(61.) *Involve both the coefficients of the radical, and the radical itself, to the required power.*

Thus, $(3\sqrt{a})^3 = 9\sqrt{a^3}$, and by reducing, $= 9a\sqrt{a}$;

and, $(10\sqrt[3]{3a^2b})^2 = 100\sqrt[3]{9a^4b^2} = 100a\sqrt[3]{9ab^2}$.

This rule results directly from the rul[illegible] iplication of radicals; for $\sqrt{a} \times \sqrt{a} \times \sqrt{a} = (\sqrt{a})^3 =$ [illegible] $\sqrt{a}$.

EXAMPLES.

1. What is the 3d power of $6\sqrt[4]{4bc^2}$?

Ans. $432c\sqrt[4]{4b^3c^2}$.

2. Involve $2a\sqrt{3bc^3}$ to the 5th power.

Ans. $288a^5b^2c^7\sqrt{3bc}$.

3. Involve $bc\sqrt[3]{\frac{2}{a}}$ to the 4th power.

Ans. $\frac{2b^4c^4}{a}\sqrt[3]{\frac{2}{a}}$.

4. Involve $\sqrt{\frac{1}{2}}$ to the 4th power. Ans. $\frac{1}{4}$.

When the index of the radical is divisible by the index of the required power, it is only necessary to divide the index of the radical by the index of the required power; for, $(\sqrt[4]{a})^2 = \sqrt[4]{a^2}$, which is the same as $a^{\frac{2}{4}} = a^{\frac{1}{2}} = \sqrt{a}$.

5. Involve $\sqrt[6]{3a^2b}$ to the 3d power.

Ans. $\sqrt{3a^2b} = a\sqrt{3b}$.

6. Involve $\sqrt[4]{2b^3c}$ to the 2d power.

Ans. $\sqrt{2b^3c} = b\sqrt{2bc}$.

7. Involve $\sqrt[9]{125}$ to the 3d power. Ans. $\sqrt[3]{125} = 5$.

EVOLUTION OF RADICALS.

RULE.

(62.) *Extract the root of the coefficient, and also of the quantity under the radical sign: but if this cannot be done, put the coefficient, if there be any, under the radical sign* (56), *and multiply the index of the radical by the index of the required root.*

Thus, the square root of $9\sqrt[3]{a^2b^2} = 3\sqrt[3]{ab}$; and the cube root of $3a\sqrt{2b} = \sqrt[6]{18a^2b}$.

EXAMPLES.

1. Extract the square root of $\sqrt[n]{3ab}$. Ans. $\sqrt[2n]{3ab}$.

2. Extract the cube root of $a^3\sqrt[4]{2ac}$. Ans. $a\sqrt[12]{2ac}$.

3. Extract the 4th root of $3\sqrt{5a^3}$. Ans. $\sqrt[8]{45a^3}$.

4. Extract the square root of $2bc\sqrt{\frac{a}{2}}$.

Ans. $\sqrt[4]{2ab^2c^2}$.

5. Extract the cube root of $(a+b)\sqrt{a-b}$.

Ans. $\sqrt[6]{(a+b)^2(a-b)}$.

6. Extract the square root of $4\sqrt[3]{\frac{1}{4}}$.

Ans. $\sqrt[6]{16} = \sqrt[3]{4}$.

IMAGINARY QUANTITIES.

(63.) Since the even powers of any quantity, whether positive or negative, are positive (38), it follows that the signs of the roots of even powers may be either positive or negative. Thus, the square root of $4a^2$, may be either $+2a$, or $-2a$; for the second power of either of these quantities is $4a^2$. The same is true of the root of any even power.

This ambiguity, with respect to the sign, is expressed by the double sign ±, read *plus* or *minus*. Thus, we say, $\sqrt{a^2} = \pm a$, $\sqrt[4]{16} = \pm 2$; and if we have the equation $x^2 = 16$, we obtain, by extracting the square root of both members, $x = \pm 4$.

There is no ambiguity in the signs of the roots of the odd powers, since these powers have always the same signs as their roots (38).

From what has been said, it will appear that it is impossible to extract an even root of a negative quantity; since the even powers of any quantity, whether positive or negative, are positive, $\sqrt{-4a^2}$, $\sqrt[4]{-16}$, and $\sqrt{-15b^3}$, are, therefore, indications of operations which it is impossible to perform. They are called *imaginary quantities*, or *imaginary expr*
and may be considered as symbols of operations which
be performed.

Imaginary quantities may, however, be subjected to the operations described in the preceding rules: thus,

$\sqrt{-a}\times\sqrt{-b}=\sqrt{ab}$, which is a real quantity.

$\sqrt{-8a^2b}=\sqrt{4a^2\times-2b}=2a\sqrt{-2b}$.

$\sqrt[4]{-16}=\sqrt[4]{16\times-1}=2\sqrt[4]{-1}$.

Of Fractional Exponents.

(64.) We have hitherto expressed irrational quantities by placing them under the radical sign, but they may also be written with fractional exponents, a method which, as has been shown (45), is deduced from the rule for extracting the roots of monomials. By this system of notation, $\sqrt{a}$ is written $a^{\frac{1}{2}}$, $\sqrt[3]{a^2}$ is written $a^{\frac{2}{3}}$, and, in general, any root is expressed by a fractional exponent which has, for its numerator, the exponent of the given quantity, and for its denominator, the index of the required root. This method of notation is, on some accounts, more convenient than the other, for the rules of multiplication, division, &c., are the same for these as for monomials with integral exponents.

Multiplication of Quantities having Fractional Exponents.

Rule.

(65.) *Multiply the coefficients together, and add the exponents of the same letters:—*

Thus, $a^{\frac{2}{3}}\times a^{\frac{1}{2}}=a^{\frac{2}{3}+\frac{1}{2}}=a^{\frac{7}{6}}$.

For, $a^{\frac{2}{3}}=\sqrt[3]{a^2}$, $a^{\frac{1}{2}}=\sqrt{a}$;

and, $\sqrt[3]{a^2}\times\sqrt{a}=\sqrt[6]{a^7}=a^{\frac{7}{6}}$.

Examples.

1. Multiply $3a^{-\frac{1}{3}}$ by $2a^{\frac{2}{5}}$.

Ans. $3a^{-\frac{1}{3}}\times 2a^{\frac{2}{5}}=6a^{\frac{2}{5}-\frac{1}{3}}=6a^{\frac{1}{15}}$.

2. Multiply $2a^{-\frac{m}{n}}$ by $5a^{\frac{p}{q}}$.

$$\text{Ans. } 2a^{-\frac{m}{n}} \times 5a^{\frac{p}{q}} = 10a^{\frac{p}{q}-\frac{m}{n}} = 10a^{\frac{np-mq}{nq}}.$$

3. Multiply $(a+b)^{\frac{1}{2}}$ by $(a+b)^{\frac{1}{3}}$

Ans. $(a+b)^{\frac{5}{6}}$.

4. Multiply $a^{\frac{2}{3}}b^{\frac{3}{2}}$ by $a^{\frac{1}{3}}b^{\frac{2}{3}}$. Ans. $ab^{\frac{13}{6}}$.

5. Multiply $3a^{-\frac{1}{2}}b^2$ by $3a^{\frac{1}{2}}b^{-2}$. Ans. 9.

6. Multiply $7ax^{\frac{1}{4}}$ by $2bx^{\frac{4}{3}}$. Ans. $14abx^{\frac{19}{12}}$.

7. Multiply $6a^{\frac{3}{2}}x^{\frac{1}{2}}$ by $11a^{\frac{2}{3}}x$. Ans. $6a^{\frac{13}{6}}x^{\frac{3}{2}}$.

DIVISION OF QUANTITIES HAVING FRACTIONAL EXPONENTS.

RULE.

(66.) *Divide the coefficient of the dividend by the coefficient of the divisor, and subtract the exponents of the same letter.*

Thus, $a^{\frac{2}{3}} \div a^{\frac{1}{2}} = a^{\frac{2}{3}-\frac{1}{2}} = a^{\frac{1}{6}}$.

For, $a^{\frac{2}{3}} = \sqrt[3]{a^2}$, $a^{\frac{1}{2}} = \sqrt{a}$;

and, $\sqrt[3]{a^2} \div \sqrt{a} = \sqrt[6]{a^4} \div \sqrt[6]{a^3} = \sqrt[6]{a} = a^{\frac{1}{6}}$.

EXAMPLES.

1. Divide $a^{\frac{2}{3}}$ by $a^{-\frac{1}{2}}$.

$$\text{Ans. } a^{\frac{2}{3}} \div a^{-\frac{1}{2}} = a^{\frac{2}{3}-(-\frac{1}{2})} = a^{\frac{2}{3}+\frac{1}{2}} = a^{\frac{7}{6}}.$$

2. Divide $8a^{\frac{1}{3}}b^{\frac{2}{3}}$ by $2a^{\frac{1}{4}}b^{\frac{1}{2}}$.

$$8a^{\frac{1}{3}}b^{\frac{2}{3}} \div 2a^{\frac{1}{4}}b^{\frac{1}{2}} = 4a^{\frac{1}{3}-\frac{1}{4}}b^{\frac{2}{3}-\frac{1}{2}} = 4a^{\frac{1}{12}}$$

3. Divide $12a^{\frac{2}{3}}$ by $6a^{\frac{3}{5}}$. An

4. Divide $ab^{\frac{13}{6}}$ by $a^{\frac{1}{3}}b^{\frac{2}{3}}$. Ans. $a^{\frac{2}{3}}b^{\frac{3}{2}}$.

5. Divide $ab^{\frac{1}{2}}c^{\frac{5}{4}}$ by $ac^{\frac{3}{4}}$. Ans. $b^{\frac{1}{2}}c^{\frac{1}{2}}$.

6. Divide $26a^{-\frac{2}{3}}x^{\frac{1}{3}}$ by $2a^{\frac{1}{3}}x^{-\frac{1}{2}}$.

Ans. $13a^{-1}x^{\frac{5}{6}}$.

7. Divide $a^{\frac{m}{n}}$ by $a^{\frac{p}{q}}$. Ans. $a^{\frac{mq-np}{nq}}$.

FORMATION OF POWERS.

RULE.

(67.) *Multiply the exponent of each letter by the exponent of the required power.*

Thus, $(a^{\frac{2}{3}})^2 = a^{\frac{4}{3}}$.

For, $a^{\frac{2}{3}} = \sqrt[3]{a^2}$; and $(\sqrt[3]{a^2})^2 = \sqrt[3]{a^4} = a^{\frac{4}{3}}$.

EXAMPLES.

1. Involve $a^{\frac{3}{4}}b^{\frac{1}{2}}$ to the 2d power. Ans. $a^{\frac{3}{2}}b$.

2. Involve $4a^{\frac{1}{3}}c^{\frac{2}{3}}$ to the 3d power. Ans. $64ac^2$.

3. Involve $\frac{2a^{\frac{2}{3}}}{b^{\frac{1}{2}}}$ to the 5th power. Ans. $\frac{32a^{\frac{10}{3}}}{b^{\frac{5}{2}}}$.

4. Involve $\frac{a^{-\frac{1}{4}}b^{\frac{1}{2}}}{c^{\frac{1}{6}}}$ to the 6th power. Ans. $\frac{a^{-\frac{3}{2}}b^3}{c}$.

5. Involve $a^{\frac{m}{n}}$ to the $\frac{r}{p}$ power.

Ans. $a^{\frac{m}{n}\times\frac{r}{p}} = a^{\frac{mr}{np}}$.

7. Involve $a^{\frac{1}{2}}$ to the power $\frac{1}{3}$. Ans. $a^{\frac{1}{6}}$.

EXTRACTION OF ROOTS.

RULE.

(68.) *Extract the root of the coefficient, and divide the exponent of each letter by the index of the required root.*

Thus, the square root of $a^{\frac{4}{3}}$ is $a^{\frac{2}{3}}$; and the cube root of $a^{\frac{1}{2}}$ is $a^{\frac{1}{2}\div 3} = a^{\frac{1}{6}}$.

EXAMPLES.

1. Extract the square root of $a^{\frac{3}{4}}b^{\frac{1}{2}}$. Ans. $a^{\frac{3}{8}}b^{\frac{1}{4}}$.

2. Extract the cube root of $8a^{\frac{2}{3}}bc^{\frac{3}{2}}$. Ans. $2a^{\frac{2}{9}}b^{\frac{1}{3}}c^{\frac{1}{2}}$.

3. Extract the nth root of $a^{\frac{m}{n}}$. Ans. $a^{\frac{m}{nn}}$.

4. Extract the $\frac{m}{n}$th root of $a^{\frac{x}{p}}$.

Ans. $a^{\frac{x}{p}\times\frac{n}{m}} = a^{\frac{xn}{mp}}$.

5. Extract the square root of $25a^{-\frac{1}{2}}x^{\frac{2}{3}}$.

Ans. $5a^{-\frac{1}{4}}x^{\frac{1}{3}}$.

6. Extract the 5th root of $32a^{\frac{4}{5}}x^{\frac{5}{3}}$ Ans. $2a^{\frac{4}{25}}x^{\frac{1}{3}}$.

POLYNOMIALS HAVING RADICAL TERMS.

(69.) The same rules will apply to polynomials, some, or all the terms of which are radicals, as have been applied to polynomials in general, with this difference only, that the operations performed upon the radical terms of these polynomials, follow the rules which have been laid down in the preceding pages. It will be sufficient, therefore, to give a number of examples, for exercise, in the application of those rules.

EXAMPLES.

1. Multiply $a+\sqrt{x}$ by $a-\sqrt{x}$.

$$\begin{array}{l}
a+\sqrt{x} \\
a-\sqrt{x} \\
\hline
a^2+a\sqrt{x} \\
\quad -a\sqrt{x}+x \\
\hline
a^2-x
\end{array}$$

2. Multiply $\sqrt{a}-\sqrt[3]{x}$ by $2\sqrt{a}+\sqrt[3]{x}$

$$\begin{array}{l}
\sqrt{a}-\sqrt[3]{x} \\
2\sqrt{a}+\sqrt[3]{x} \\
\hline
2a-2\sqrt[6]{a^3x^2} \\
\qquad +\sqrt[6]{a^3x^2}-\sqrt[3]{x^2} \\
\hline
2a-\sqrt[6]{a^3x^2}-\sqrt[3]{x^2}
\end{array}$$

In this example, it became necessary to reduce the radicals, $\sqrt{a}$ and $\sqrt[3]{x}$, to a common index, before they could be multiplied.

A more simple way would be, to change the radicals into quantities with fractional exponents, and multiply according to the rules for quantities of that kind.

Thus, $\sqrt{a}-\sqrt[3]{x}=a^{\frac{1}{2}}-x^{\frac{1}{3}}$;

and, $2\sqrt{a}+\sqrt[3]{x}=2a^{\frac{1}{2}}+x^{\frac{1}{3}}$.

$$\begin{array}{l}
2a-2a^{\frac{1}{2}}x^{\frac{1}{3}} \\
\qquad +a^{\frac{1}{2}}x^{\frac{1}{3}}-x^{\frac{2}{3}} \\
\hline
\end{array}$$

Product, $2a-2a^{\frac{1}{2}}x^{\frac{1}{3}}-x^{\frac{2}{3}}$

3. Multiply $a+\sqrt{x}$ by $3b+\sqrt{2x}$.

Ans. $3ab+3b\sqrt{x}+a\sqrt{2x}+\sqrt{2x^2}$.

or, $3ab+3b\sqrt{x}+a\sqrt{2x}+x\sqrt{2}$.

4. Multiply $\sqrt{a}+\sqrt{a+x}$ by $2\sqrt{3a+3x}$.

Ans. $2\sqrt{3a^2+3ax}+2(a+x)\sqrt{3}$.

5. Multiply $2\sqrt{\frac{1}{2}}+2\sqrt{2}$ by $4\sqrt{\frac{1}{8}}+2\sqrt{8}$. Ans. 30.

6. Divide $a-x$ by $\sqrt{a}-\sqrt{x}$.

$$\begin{array}{l|l}
a-x & \sqrt{a}-\sqrt{x} \\
\underline{a-\sqrt{ax}} & \overline{\sqrt{a}+\sqrt{x}}, \text{ the answer.} \\
\quad +\sqrt{ax}-x & \\
\quad \underline{+\sqrt{ax}-x} &
\end{array}$$

7. Divide $6\sqrt{a^3}-2\sqrt{ax}$ by $2\sqrt{a}$. Ans. $3a-\sqrt{x}$.

8. Divide $3ab+3b\sqrt{x}+a\sqrt{2x}+\sqrt{2x^2}$ by $a+\sqrt{x}$.

Ans. $3b+\sqrt{2x}$.

9. Involve $\sqrt{a}-\sqrt{x}$ to the 2d power.

Ans. $a-2\sqrt{ax}+x$.

10. Involve $a+\sqrt[3]{x}$ to the 3d power.

Ans. $a^3+3a^2\sqrt[3]{x}+3a\sqrt[3]{x^2}+x$.

11. Involve $\sqrt{3a}+\sqrt{2x}$ to the 2d power.

Ans. $3a+\sqrt{6ax}+2x$.

12. Involve $2\sqrt{\frac{1}{3}}-3\sqrt{\frac{1}{12}}$ to the 2d power.

Ans. $4\times\frac{1}{3}-12\times\frac{1}{6}+9\times\frac{1}{12}=\frac{1}{12}$.

13. Extract the square root of $9a+36\sqrt{3ax}+108x$.

$$\begin{array}{r|l} & \text{The Root.} \\ & 9a+36\sqrt{3ax}+108x \;\|\; 3\sqrt{a}+6\sqrt{3x} \\ & 9a \\ \hline 6\sqrt{a}+6\sqrt{3x} & 36\sqrt{3ax}+108x \\ & 36\sqrt{3ax}+108x \\ \hline \end{array}$$

14. Extract the cube root of $a^3+3a^2\sqrt[3]{x}+3a\sqrt[3]{x^2}+x$.

$$\begin{array}{r|l} & a^3+3a^2\sqrt[3]{x}+3a\sqrt[3]{x^2}+x \;\|\; a+\sqrt[3]{x} \\ & a^3 \\ \hline 3a^2-3a\sqrt[3]{x}+\sqrt[3]{x^2} & 3a^2\sqrt[3]{x}+3a\sqrt[3]{x^2}+x \\ & 3a^2\sqrt[3]{x}+3a\sqrt[3]{x^2}+x \\ \hline \end{array}$$

15. Extract the square root of $a-2\sqrt{ax}+x$.

Ans. $\sqrt{a}-\sqrt{x}$.

16. Extract the square root of $a^2+4a\sqrt{ax}+6ax+4x\sqrt{ax}+x^2$.

Ans. $a+2\sqrt{ax}+x$.

EQUATIONS INVOLVING RADICAL QUANTITIES.

(70.) An equation may contain a root of the unknown quantity instead of that quantity itself; and in such a case, the rules given for the reduction of simple equations are not sufficient. The equation $\sqrt{x}+a=3b-c^2$ is one of this kind. For the resolution of equations, containing a root of the unknown quantity, we give the following

RULE.

1. *Transpose the terms of the equation, so that the radical may stand alone on one side of the sign of equality.*

2. *Then involve both members of the equation to the power denoted by the index of the radical: the resulting equation will contain no radical, and may be reduced by the rules already given.*

This rule depends upon the principle, that if two quantities are equal, any powers of these quantities will also be equal; and hence the equation will not be destroyed by involving both members to the same power.

EXAMPLES.

1. Given $2\sqrt{x} + a = 2b - 2a$.

Transposing *a* we have,

$$2\sqrt{x} = 2b - 3a;$$

and squaring both members,

$$4x = 4b^2 - 12ab + 9a^2;$$

whence, $x = b^2 - 3ab + \frac{9a^2}{4}$.

2. Given $\sqrt{4 + x} = 4 - \sqrt{x}$, to find x.

Squaring both members of this equation, we have,

$$4 + x = 16 - 8\sqrt{x} + x.$$

Transposing and reducing,

$$8\sqrt{x} = 12;$$

squaring both members again,

$$64x = 144$$

$$x = \tfrac{144}{64} = \tfrac{9}{4} = 2\tfrac{1}{4}.$$

3. Given $\sqrt{x} + \sqrt{2a + x} = \frac{4a}{\sqrt{2a + x}}$, to find x.

Clearing this equation of fractions, we have,

$$\sqrt{2ax + x^2} + 2a + x = 4a;$$

and transposing, $\sqrt{2ax + x^2} = 2a - x$.

Squaring both members,

$$2ax + x^2 = 4a^2 - 4ax + x^2.$$

Transposing and reducing,

$$6ax = 4a^2;$$

whence, $x = \frac{4a^2}{6a} = \frac{4}{6}a = \frac{2}{3}a$.

4. Given $\sqrt{4a^2+x^2} = \sqrt[4]{4b^2+x^4}$, to find x.

Squaring both members we have,

$$4a^2+x^2 = \sqrt{4b^2+x^4}.$$

Squaring both members a second time,

$$16a^4+8a^2x^2+x^4 = 4b^2+x^4.$$

Transposing and reducing,

$$2a^2x^2 = b^2-4a^4;$$

whence, $\quad x^2 = \dfrac{b^2-4a^4}{2a^2};$

and extracting the square root of both members,

$$x = \sqrt{\frac{b^2-4a^4}{2a^2}} = \frac{1}{2a}\sqrt{2b^2-8a^4}.$$

5. Given $x = \sqrt{a^2+x\sqrt{b^2+x^2}} - a$, to find x.

Transposing, $\quad x+a = \sqrt{a^2+x\sqrt{b^2+x^2}}$

Squaring both members,

$$x^2+2ax+a^2 = a^2+x\sqrt{b^2+x^2};$$

or, $\quad x^2+2ax = x\sqrt{b^2+x^2}.$

Dividing by x, $\quad x+2a = \sqrt{b^2+x^2}.$

Squaring again, $\quad x^2+4ax+4a^2 = b^2+x^2;$

whence, $\quad 4ax = b^2-4a^2;$

and, $\quad x = \dfrac{b^2-4a^2}{4a} = \dfrac{b^2}{4a} - a.$

6. Given $\sqrt[3]{a^3+x^2-2ax} = \dfrac{a+x}{\sqrt[3]{a+x}}.$

Cubing both members, we have

$$a^3+x^2-2ax = \frac{(a+x)^3}{a+x} = (a+x)^2 = a^2+2ax+x^2$$

Transposing and reducing,

$$-4ax = a^2-a^3;$$

or, $\quad 4ax = a^3-a^2;$

and, $\quad x = \dfrac{a^3-a^2}{4a} = \dfrac{a^2-a}{4}.$

7. Given $\sqrt{\frac{5x}{3}} + 12 = 17$, to find x. $x = 15$.

8. Given $\sqrt{12 + x} = 2 + \sqrt{x}$, to find x. $x = 4$.

9. Given $\sqrt{x + 20} = \frac{25}{\sqrt{x + 20}}$, to find x. $x = 5$.

10. Given $\sqrt[3]{64 + x^2 - 8x} = \frac{4 + x}{\sqrt[3]{4 + x}}$, to find x. $x = 3$.

11. Given $\sqrt{5 + x} + \sqrt{x} = \frac{15}{\sqrt{5 + x}}$, to find x. $x = 4$.

12. Given $\sqrt{x + \sqrt{x}} - \sqrt{x - \sqrt{x}} = \frac{3}{2}\sqrt{\frac{x}{x + \sqrt{x}}}$, to find x.

$x = \frac{25}{16}$.

CHAPTER IV.

EQUATIONS OF THE SECOND DEGREE.

(71.) An equation of the form $x^2 = a$, or $ax^2 + bx = c$, containing the unknown quantity in the second power, is termed an equation of the second degree. An equation which contains x in no other power besides the second, as $\frac{x^2}{2} + x^2 = ab$, is called *an equation of two terms*, for it can always be reduced to such a form, that the square of the unknown quantity may constitute one member of the equation, and the known quantities the other. $\frac{x^2}{2} + x^2 = ab$ may be reduced to $x^2 = \frac{2ab}{3}$ and $\frac{x^2}{c} + \frac{x^2}{a} - 1 = 3b^2 - a$, becomes by reduction, $x^2 = \frac{3ab^2c - a^2c + ac}{a + c}$.

When the equation contains both the first and second powers of the unknown quantity, it is called an *equation of three terms*; because there must always be two terms containing the unknown quantity, and at least one, composed of known quantities. Equations of these terms can always be reduced to the form $a^2 \pm px = q$; that is, to such a form, that the first member shall consist of the second power of x, and another term, which contains the first power of x: thus, the equation $ax^2 + \frac{x}{a} - ax = a^3b^2$ can be reduced to $x^2 + \left(\frac{1-a^2}{a^2}\right)x = a^3b^2$, and $x + \frac{2}{x} + \frac{x}{2} = 5\frac{3}{4} + \frac{3}{x}$, can be reduced to $x^2 - \frac{23x}{6} = \frac{4}{6}$.

Equations of two terms are also called *incomplete equations*, and equations of three terms, *complete equations.*

Incomplete Equations of the Second Degree.

(72.) If we have the equation $ax^2 = b$, by dividing by a, it becomes $x^2 = \frac{b}{a}$; and in order to find the value of x, we have only to extract the square root of both members of the equation, and we have $x = \sqrt{\frac{b}{a}}$.

This value of x may be either positive or negative; for if we put $\frac{b}{a} = p$, we shall have $x = \sqrt{p}$; but, $\sqrt{p} \times \sqrt{p} = p$, and $-\sqrt{p} \times -\sqrt{p}$ also is equal to $+p$.

We may, therefore, suppose $x = +\sqrt{p}$, or $x = -\sqrt{p}$, since, in both instances, by squaring both members of the equation, we have $x^2 = p$.

To denote this ambiguity in the sign, the value of x is always affected with the sign ± (63).

The following rule for the solution of incomplete equations can now be understood.

Transpose the terms of the equation, so that those which contain x², *may constitute one member of the equation, and the known quantities the other; then divide by the coeffi- of* x², *and extract the square root of both members.*

EXAMPLES.

1. Given $3x^2-7=x^2+1$, to find x.

Transposing, $\qquad 2x^2=8$.

Dividing by 2, $\qquad x^2=4$;

and extracting the root, $\qquad x=\pm 2$.

2. Given $5x^2-27=3x^2+215$, to find x.

Transposing, $\qquad 5x^2-3x^2=215+27$;

or, $\qquad 2x^2=242$,

$$x^2=121,$$

$$x=\pm\sqrt{121}=11.$$

3. Given $\dfrac{a}{x}+\dfrac{\sqrt{a^2-x^2}}{x}=\dfrac{x}{b}$ to find x.

Clearing of fractions,

$$ab+b\sqrt{a^2-x^2}=x^2.$$

Transposing, $\qquad b\sqrt{a^2-x^2}=x^2-ab$.

Squaring both members,

$$b^2a^2-b^2x^2=x^4-2abx^2+a^2b^2.$$

Reducing and dividing by x^2,

$$-b^2=x^2-2ab.$$

Transposing and changing the signs,

$$x^2=2ab-b^2.$$

Extracting the root, $\qquad x=\sqrt{2ab-b^2}$.

4. Given $\dfrac{x^2}{3}+4=\dfrac{x^2}{2}+2$, to find x.

$$x=\pm\sqrt{12}=2\sqrt{3}.$$

5. Given $5x^2-1=244$. $\qquad x=7$.

6. Given $\dfrac{4x^2+5}{9}=45$. $\qquad x=10$.

7. Given $2ax^2+b-4=cx^2-5+d-ax^2$.

$$x=\sqrt{\frac{d-b-}{3a-}}$$

8. Given $\dfrac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b$, to find x.

Clearing this equation of fractions, we have,

$$a-\sqrt{a^2-x^2}=b\times(a+\sqrt{a^2-x^2});$$

and multiplying both members by $a-\sqrt{a^2-x^2}$ (17),

$$(a-\sqrt{a^2-x^2})^2=b\times(a^2-(a^2+x^2)=bx^2.$$

Extracting the square root,

$$a-\sqrt{a^2-x^2}=\pm x\sqrt{b}.$$

Transposing, $a\pm x\sqrt{b}=\sqrt{a^2-x^2};$

and squaring both members,

$$a^2\pm 2ax\sqrt{b}+x^2b=a^2-x^2.$$

Transposing, $bx^2+x^2=2ax\sqrt{b};$

or, $(b+1)x^2=2ax\sqrt{b}.$

Dividing by x, $(b+1)x=2a\sqrt{b},$

$$x=\frac{2a\sqrt{b}}{b+1}.$$

9. Given $\left.\begin{array}{r} x+y:x=5:3 \\ xy=6 \end{array}\right\}$ to find the values of x and y.

From the first we obtain,

$$3x+3y=5x, \text{ or } 3y=2x;$$

whence, $x=\dfrac{3y}{2}.$

Substituting this value in the 2d equation, we have,

$$y\times\frac{3y}{2}=6,$$

$$3y^2=12,$$

$$y^2=4,$$

$$y=2,$$

and, $x=\dfrac{3y}{2}=\dfrac{6}{2}=3.$

10. Given $\dfrac{\sqrt{x}+\sqrt{x-a}}{\sqrt{x}-\sqrt{x-a}}=\dfrac{b^2a}{x-a}$.

Multiplying both numerator and denominator of the first fraction by $\sqrt{x}-\sqrt{x-a}$, the equation becomes (17),

$$\frac{x-(x-a)}{(\sqrt{x}-\sqrt{x-a})^2}=\frac{b^2a}{x-a};$$

Reducing and dividing both members of the equation by a, we have,

$$\frac{1}{(\sqrt{x}-\sqrt{x-a})^2}=\frac{b^2}{x-a}.$$

Extracting the square root,

$$\frac{1}{\sqrt{x}-\sqrt{x-a}}=\frac{\pm b}{\sqrt{x-a}};$$

and clearing of fractions,

$$\sqrt{x-a}=\pm b(\sqrt{x}-\sqrt{x-a}.$$

Multiplying by $\sqrt{x-a}$,

$$x-a=\pm b(\sqrt{x^2-ax}-x+a;$$

or, $$x-a=\pm b\sqrt{x^2-ax}-bx+ba;$$

whence, $(x-a)+b(x-a)$, or $(1+b)(x-a)=b\sqrt{x^2-ax}$.

Squaring both members,

$$(1+b)^2(x-a)^2=b^2\times(x^2-ax)=b^2x(x-a).$$

Dividing by $x-a$,

$$(1+b)^2(x-a)=b^2x;$$

or, $$(1+b)^2x-(1+b)^2a=b^2x.$$

Transposing, $(1+b)^2x-b^2x=(1+b)^2a$;

$$(1+2b+b^2)x-b^2x;\text{ or, }(1+2b)x=(1+b)^2a;$$

whence, $$x=\frac{(1+b)^2a}{1+2b}.$$

11. Given $\sqrt[3]{x^3-a^3}=\sqrt[3]{3ax^2-3a^2x+8b}$, to find x.

Cubing both members,

$$x^3-a^3=3ax^2-3a^2x+8b.$$

Transposing, $x^3 - 3ax^2 + 3a^2x - a^3 = 8b$.
Extracting the cube root,

$$x - a = 2\sqrt[3]{b}$$

and, $$x = 2\sqrt[3]{b} + a.$$

12. Given $x + y : x - y = 3 : 1,$ and, $x^3 - y^3 = 56,$ $\Big\}$ to find x and y.

Ans. $x = 4,\ y = 2.$

13. Given $\dfrac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = b$, to find x.

$$x = \frac{2ab}{b^2+1}.$$

14. Given $x^2y + xy^2 = 30,$ $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6}$ $\Big\}$ to find x and y.

$x = 3,\ y = 2.$

15. Given $\dfrac{\sqrt{4x+1} + \sqrt{4x}}{\sqrt{4x+1} - \sqrt{4x}} = 9$, to find x. $\quad x = \dfrac{4}{9}.$

16. Given $a^2 - 2ax + x^2 = b$, to find x.

$x = a - \sqrt{b}.$

17. Given $\dfrac{1}{1 - \sqrt{1-x^2}} - \dfrac{1}{1 + \sqrt{1-x^2}} = \dfrac{\sqrt{3}}{x^2}$, to find x.

$x = \pm\frac{1}{2}.$

18. Given $\dfrac{4}{x^2 - 2x + 1} = \frac{1}{4}$, to find x. $\quad x = 5.$

19. Given $\dfrac{x^2}{9} + \dfrac{x}{3} + \dfrac{1}{4} = 12$, to find x.

$$x = -\frac{3}{2} \pm 6\sqrt{3}.$$

20. Given $2x^2 - 5 = \sqrt{x^2 + 25}$, to find x. $\quad x = \frac{1}{2}\sqrt{21}.$

21. Given $\sqrt{x^2 + 8} = \sqrt{125 - 6x^2 - 12x}$, to find x.

$x = 3.$

COMPLETE EQUATIONS OF THE SECOND DEGREE.

(73.) The rules which have been given will not be sufficient for the resolution of complete equations of the second degree.

For example; the equation $x^2 \pm px = q$, in which p and q are supposed to represent any numbers whatever, whether integral or fractional, cannot be reduced by extracting the root of both members; for the first member, both the terms of which contain the unknown quantity, is not a perfect square. Indeed, it is evident, that there can be no exact root of a binomial; for the square of a monomial is a single term, and the square of a binomial consists of three terms (41), the square of the first, twice the product of the first and second, and the square of the second.

Now, if in the equation $x^2 \pm px = q$, we add another term to each member, such as will make the first a trinomial, and a perfect square, the equation will not be destroyed. To discover what this term should be, let it be observed that $(x+a)^2 = x^2 + 2ax + a^2$, and $(x-a)^2 = x^2 - 2ax + a^2$; or, uniting these expressions into one by means of the double sign, ±, $(x \pm a)^2 = x^2 \pm 2ax + a^2$. Now, if we had the expression $x^2 \pm 2ax$, it is evident, that we should make it a perfect square by adding a^2, which is the square of half the coefficient of x in the second term. In like manner, we may make $x^2 \pm px$, a perfect square by adding the square of $\frac{1}{2}p$; p being the coefficient of x in the second term. We shall then have $x^2 \pm px + \frac{p^2}{4}$, the square root of which is $x \pm \frac{p}{2}$ (41).

In order, therefore, to reduce the equation $x^2 \pm px = q$, we add $\frac{p^2}{4}$ to both members; the equation then becomes,

$$x^2 \pm px + \frac{p^2}{4} = q + \frac{p^2}{4};$$

and extracting the root,

$$x \pm \frac{p}{2} = \pm \sqrt{q + \frac{p^2}{4}};$$

whence, $$x = \mp \frac{p}{2} \pm \sqrt{q + \frac{p}{4}}.$$

The double sign is used before px in this equation, inmuch as the same demonstration will apply, whether

$x^2 - px = q$, or $x^2 + px = q$. In the root, the sign of the last term corresponds to the sign of px.

From this reasoning, we deduce the following rule for the reduction of complete equations of the second degree.

RULE.

(74.) *Bring the equation to the form* x² ± px = q, *that is, so that the first member may consist of* x², *and another term containing the first power of* x. *Then add the square of one half the coefficient of* x *to both members, and extract the square root. The equation will then be reduced to a simple equation, which may be solved by the foregoing rules.*

EXAMPLES.

1. Given $ax^2 - bx + c = cx^2 + 2c$, to find x.

By transposition, $ax^2 - cx^2 - bx = c$;

or, $(a-c)x^2 - bx = c$.

Dividing by the coefficient of x^2,

$$x^2 - \frac{bx}{a-c} = \frac{c}{a-c}.$$

Here $\frac{b}{a-c}$ is the coefficient of x, and $\frac{1}{2}$ of it is $\frac{b}{2a-2c}$: squaring this, therefore, and adding to it both members, we have,

$$x^2 - \frac{bx}{a-c} + \frac{b^2}{(2a-2c)^2} = \frac{c}{a-c} + \frac{b^2}{(2a-2c)^2};$$

and extracting the root,

$$x - \frac{b}{2a-2c} = \pm\sqrt{\frac{c}{a-c} + \frac{b^2}{(2a-2c)^2}};$$

whence, $x = \frac{b}{2a-2c} \pm \sqrt{\frac{c}{a-c} + \frac{b^2}{(2a-2c)^2}}$.

2. Given $3x^2 - 30 = 4x + 2$.

Transposing, $3x^2 - 4x = 32$.

Dividing by 3, $x^2 - \frac{4x}{3} = \frac{32}{3}$.

Here, the coefficient of x is $\frac{4}{3}$; the half of it is $\frac{2}{3}$; squaring this, we have, $\frac{4}{9}$, to add to each member, and the equation becomes,
$$x^2-\frac{4x}{3}+\frac{4}{9}=\frac{32}{3}+\frac{4}{9}.$$
Extracting the root,
$$x-\frac{2}{3}=\pm\sqrt{\frac{32}{3}+\frac{4}{9}}=\pm\sqrt{\frac{96}{9}+\frac{4}{9}}$$
$$=\pm\sqrt{\frac{100}{9}}=\pm\frac{10}{3};$$
whence, $x=\frac{2}{3}\pm\frac{10}{3}=\frac{12}{3}=4$; or by taking — for the sign of $\frac{10}{3}$; $x=\frac{2}{3}-\frac{10}{3}=-\frac{8}{3}$.

Either of these values of x will satisfy the equation, if substituted with their proper signs; for, putting 4, the first value, in the equation, in the place of x, we have,
$$3(4)^2-30=4\times4+2;$$
or, $$48-30=16+2,$$
which gives $$48=48.$$
Substituting $-\frac{8}{3}$, we have,
$$3\left(-\frac{8}{3}\right)^2-30=4\left(-\frac{8}{3}\right)+2;$$
and performing the operations indicated,
$$3\left(\frac{64}{9}\right),\text{ or }\frac{64}{3}-30=-\frac{32}{3}+2;$$
or, $$64-90=-32+6;$$
whence, $$64+32=90+6,$$
or, $$96=96.$$

3. Given $x^2-x=-\frac{1}{4}$, to find x.
Completing the square,
$$x^2-x+\frac{1}{4}=-\frac{1}{4}+\frac{1}{4}=0.$$

Extracting the root, $x - \frac{1}{2} = 0$;

whence, $x = \frac{1}{2}$.

(75.) If we take the equation $ax^2 + bx = c$, and multiply both members by $4a$, it becomes,

$$4a^2x^2 + 4abx = 4ac;$$

then adding b^2 to each member, we have,

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2.$$

The first member of this last equation is a perfect square, the middle term being twice the product of the square roots of the extreme terms (41). We can, therefore, extract the root of both members, and the equation will be reduced to

$$2ax + b = \pm\sqrt{4ac + b^2}.$$

Transposing, $2ax = \pm\sqrt{4ac + b^2} - b$;

and dividing by $2a$, $x = \pm\frac{\sqrt{4ac + b^2} - b}{2a}$.

Now, as any equation may be reduced to the form $ax^2 \pm bx = c$, we have another rule for the reduction of complete equations of the second degree.

Rule II.

Bring the equation to the form $ax^2 \pm bx = c$; *that is, so that the first member has two terms, one of which contains* x^2, *and the other* x. *Then multiply both members by four times the coefficient of* x^2, *and add the square of the coefficient of* x *to both; and extract the square root. The resulting equation may be reduced by the rules for simple equations.*

Examples.

1. Given $2x^2 - 6x = 80$, to find x.

Multiplying by 8, which is four times the coefficient of x^2, we have, $16x^2 - 48x = 640$.

Adding the square of the coefficient of x,

$$16x^2 - 48x + 36 = 640 + 36;$$

whence, $4x - 6 = \pm\sqrt{640+36} = \pm\sqrt{676} = \pm 26$,

$$4x = 6 \pm 26 = 32, \text{ or } -20,$$

and, $x = 8$, or -5.

2. Given $x^2 + 3x = 72$, to find x.

Multiplying by 4, $4x^2 + 12x = 288$.

Adding the square of the coefficient of x,

$$4x^2 + 12x + 9 = 288 + 9 = 297.$$

Extracting the root,

$$2x + 3 = \pm\sqrt{297} = 3\sqrt{33}.$$

Dividing by 2, $x + \frac{3}{2} = \frac{3}{2}\sqrt{33}.$

whence, $x = -\frac{3}{2} \pm \frac{3}{2}\sqrt{33}.$

3. Given $\frac{8 - x}{2} - \frac{2x - 11}{x - 3} = \frac{x - 2}{6}$, to find x.

Clearing of fractions,

$$33x - 3x^2 - 72 - 12x + 66 = x^2 - 5x + 6.$$

Transposing and reducing,

$$26x - 4x^2 = 12.$$

Changing the signs and dividing by 2,

$$2x^2 - 13x = -6.$$

Then by Rule II.

$$16x^2 - 104x + 169 = -48 + 169 = 121.$$

Extracting the root, $4x - 13 = \pm 11$,

or, $4x = 24$, or 2,

and, $x = 6$, or $\frac{1}{2}$.

4. Given $\sqrt{x + 5} \times \sqrt{x + 12} = 12$, to find x.

Squaring both members,

$$(x + 5) \times (x + 12) = 144;$$

or, $x^2 + 17x = 84$;

Then by Rule II,

$$4x^2 + 68x + (17)^2 = 336 + 289 = 625.$$

Extracting the root,

$$2x + 17 = \pm\sqrt{625} = 25;$$

therefore, $2x = -17 \pm 25 = 8$; or -42;

and, $x = 4$, or -21.

NOTE.—The rules given for reducing equations of the form $x^2 \pm px = q$, will apply to all equations which contain x in two powers, one of which is double the other. The following examples will show their application to a variety of such equations:—

5. Given $x^{2n}+px^n=q$, to find x.

Adding the square of $\frac{1}{2}p$ to both members, we have,

$$x^{2n}+px^n+\frac{p^2}{4}=q+\frac{p^2}{4};$$

Extracting the square root,

$$x^n+\frac{p}{2}=\pm\sqrt{q+\frac{p^2}{4}};$$

whence, $$x^n=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}};$$

and, $$x=\sqrt[n]{-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}}.$$

By Rule II, $4x^{2n}+4px^n+p^2=4q+p^2$.

Extracting the root,

$$2x^n+p=\pm\sqrt{4q+p^2};$$

whence, $$x^n=-\frac{p}{2}\pm\frac{\sqrt{4q+p^2}}{2}=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}};$$

and, $$x=\sqrt[n]{-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}}.$$

6. Given $x^6-4x^3=621$, to find the value of x.

Completing the square,

$$x^6-4x^3+4=621+4=625.$$

Extracting the square root,

$$x^3-2=\pm\sqrt{625}=25;$$

whence, $$x^3=2+25=27.$$

Extracting the cube root, $x=3$.

7. Given $\sqrt{x+a}+b\sqrt[4]{x+a}=2b^2$, to find x.

Completing the square,

$$\sqrt{x+a}+b\sqrt[4]{x+a}+\frac{b^2}{4}=2b+\frac{b^2}{4};$$

Extracting the square root,

$$\sqrt[4]{x+a}+\frac{b}{2}=\pm\sqrt{2b^2+\frac{b^2}{4}}=\pm\sqrt{\frac{9b^2}{4}}=\pm\frac{3b}{2};$$

therefore, $\sqrt[4]{x+a}=-\frac{b}{2}\pm\frac{3b}{2}=\frac{2b}{b}=b.$

Raising both numbers to the fourth power,

$$x+a=b^4;$$

whence, $x=b^4-a.$

Examples for Exercise.

1. Given $x^2+12x=108$. $x=6$.
2. Given $x^2-14x=51$. $x=17$.
3. Given $x^2+6bx=c^2$. $x=\pm\sqrt{c^2+9b^2}-3b$.
4. Given $2x^2-5x=117$. $x=9$.
5. Given $7x^2-20x=32$. $x=4$.
6. Given $5x^2+4x=273$. $x=7$.
7. Given $\frac{x^2}{6}-1=x+11$. $x=12$.
8. Given $\frac{6}{x+1}+\frac{2}{x}=3$. $x=2$.
9. Given $x^2-34=\frac{1}{3}x$. $x=6$.
10. Given $x^2-6x+19=13$. $x=4.732$, or 1.268.
11. Given $\frac{x}{5}+\frac{5}{x}=5\frac{1}{5}$. $x=25$, or 1.
12. Given $\frac{x}{a}+\frac{a}{x}=\frac{2}{a}$ $x=1\pm\sqrt{1-a^2}$.
13. Given $\frac{3x}{2}+\frac{2}{3x}=x+\frac{2x-2}{3}$. $x=2\pm2\sqrt{2}$.
14. Given $\sqrt{x+12}+\sqrt[4]{x+12}=6$. $x=4$.
15. Given $x^3-x^{\frac{3}{2}}=56$. $x=4$.
16. Given $6x+\frac{35-3x}{x}=44$. $x=7$, or $\frac{5}{6}$.
17. Given $\frac{16}{x}-\frac{100-9x}{4x^2}=3$. $x=4$, or $\frac{25}{12}$.

18. Given $\frac{a-x}{x}+\frac{x}{a-x}=\frac{b}{c}$.

$$x=\frac{a}{2}\pm\frac{a}{2}\sqrt{\frac{b-2c}{b+2c}}, \text{ or } \frac{a}{2}\pm\frac{a}{2b+4c}\sqrt{b^2-4c^2}.$$

19. Given $\sqrt{9x+4}=3x$. $x=\frac{4}{3}$, or $\frac{1}{3}$.

20. Given $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$. $x=4$, or $\frac{64}{9}$.

21. Given $\frac{\sqrt{a^2x}+b}{a+\sqrt{x}}=\frac{a-\sqrt{x}}{\sqrt{x}}$.

$$x=\left(\frac{-b\pm\sqrt{4a^3+4a^2+b^2}}{2(a+1)}\right)^2.$$

QUESTIONS PRODUCING EQUATIONS OF THE SECOND DEGREE.

(77.) Divide 34 into two such parts, that their product shall be 225.

Let $x=$ one of the parts; then will the other be $34-x$; and taking their product, we have,

$$x(34-x)=225,$$

or,
$$34x-x^2=225;$$

or by changing the signs,

$$x^2-34x=-225.$$

Completing the square,

$$x^2-34x+289=-225+289=64;$$

extracting the root,

$$x-17=\pm\sqrt{64}=\pm 8;$$

therefore, $x=17\pm 8=25$, or 9.

In this case, the two values of x give the two parts into which the given number is to be divided, for, $25\times 9=225$.

2. There are two numbers whose difference is 7, and half their product plus 30, is equal to the square of the lesser number. What are the numbers?

Let x be the less; then $x+7$ will be the greater; and by the conditions of the question.

$$\frac{x(x+7)}{2}+30=x^2.$$

Multiplying and clearing of fractions,

$$x^2+7x+60=2x^2;$$

transposing, $x^2-7x=60$.

Completing the square by Rule II.

$$4x^2-28x+49=240+49=289;$$

extracting the root, $2x-7=\pm 17$;

transposing, $2x=7\pm 17=24$, or -10;

$$x=12, \text{ or } -5.$$

12 is therefore the less, and $12+7=19$, the greater; or, if we take the other value of x, which is -5, we shall have for the greater, $-5+7=2$. Either of these values will satisfy the conditions of the equation, for

$$\frac{19\times 12}{2}+30=144, \text{and } \frac{-5\times 2}{2}+30=25.$$

3. Divide the number 30 into two such parts, that their product shall be equal to eight times their difference.

Let $x=$ the lesser part; then $30-x$ will be the greater, and their difference will be

$$(30-x)-x=30-2x.$$

Then, by the conditions of the problem,

$$x(30-x)=8(30-2x);$$

or, $30x-x^2=240-16x$.

Transposing and changing the signs,

$$x^2-46x=-240.$$

Completing the square by Rule I.

$$x^2-46x+529=-240+529=289;$$

extracting the root, $x-23=\pm 17$;

whence, $x=23\pm 17=6$, or 40.

Of these values of x, it is evident, that only one will satisfy the conditions of the question, for 40 cannot be the lesser part, nor any part of 30; 6 is therefore the value of x, and we have for the greater part, $30-6$, or 24, which answers the conditions of the question, for

$$6\times 24=144=(30-6)-6)8, \text{ or } 18\times 8=144.$$

4. A person bought a number of sheep for \$120. If there had been 8 more, each sheep would have cost him half a dollar less. What was the cost of a sheep?

Let $x =$ the number of sheep, then $\frac{120}{x}$ is the price of one sheep; but if he had bought 8 more for the same money, the price of each sheep would have been $\frac{120}{x+8}$; and by the conditions, the difference between these two prices is half a dollar; we have, therefore,

$$\frac{120}{x} - \frac{120}{x+8} = \frac{1}{2}.$$

Clearing of fractions,

$$240x + 1920 - 240x = x^2 + 8x.$$

Transposing and reducing,

$$x^2 + 8x = 1920.$$

Completing the square,

$$x^2 + 8x + 16 = 1920 + 16 = 1936.$$

Extracting the root, $x + 4 = \pm 44$;

whence, $x = -4 \pm 44 = 40$, or -48.

The value found for x, by making 44 positive, answers the required conditions; and the other value, -48, would be the answer to another problem, which is thus enunciated:—*A person bought a number of sheep for* \$120. *If there had been* 8 *less, each sheep would have cost him half a dollar more.*

NOTE.—It may sometimes occur, that after completing the square, the second member of the equation is negative, in which case, the root cannot be extracted (63). Such a result is generally an indication of some impossibility in the conditions from which the equation is derived, or an error in forming the equation. If we have the equation $x^2 - 2x = -13$, by completing the square, and reducing, we shall have

$$x = 1 \pm \sqrt{-12},$$

in which the value of x contains an imaginary expression.

For another example, let it be required to divide 20 into two such parts, that their product shall be 125.

Let $x =$ one part, then $20 - x$ will be the other, and

$$x(20 - x) = 125,$$

or, $20x - x^2 = 125.$

Transposing, $x^2 - 20x = -125.$

Completing the square,

$$x^2 - 20x + 100 = -125 + 100 = -25;$$

and, $x - 10 = \pm\sqrt{-25},$

or, $x = 10 \pm \sqrt{-25}.$

Here, the imaginary expression which enters into the value of x, shows an impossibility in the required conditions. Indeed it is not possible to divide 20 into two parts, the product of which shall be 125, for the greatest product that can be formed of the parts of 20, is 100, the product of 10 by 10.

5. The sum of two numbers is 8, and the sum of their 4th powers 1312; what are those numbers?

Let $x =$ half the difference of those numbers, and for the sake of brevity, put $8 = 2a$, and $1312 = b$.

The greater number will be $a+x$, and the lesser $a-x$ (17); then we shall have the equation,

$$\left.\begin{matrix} (a+x)^4 \\ +(a-x)^4 \end{matrix}\right\} = b.$$

$$\text{or,} \left\{\begin{matrix} a^4 + 4a^3x + 6a^2x^2 + 6ax^3 + x^4 \\ +a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4 \end{matrix}\right\} = b,$$

and reducing, $2a^4 + 12a^2x^2 + 2x^4 = b.$

Dividing by 2, $a^4 + 6a^2x^2 + x^4 = \frac{1}{2}b;$

transposing, $x^4 + 6a^2x^2 = \frac{1}{2}b - a^4.$

Completing the square,

$$x^4 + 6a^2x + 9a^4 = \tfrac{1}{2}b - a^4 + 9a^4 = \tfrac{1}{2}b + 8a^4.$$

Extracting the square root,

$$x^2 + 3a^2 = \pm\sqrt{\tfrac{1}{2}b + 8a^4};$$

$$x^2 = -3a \pm \sqrt{\tfrac{1}{2}b + 8a^4}.$$

Extracting the square root,

$$x = \pm\sqrt{-3a^2 \pm \sqrt{\tfrac{1}{2}b + 8a^4}}.$$

Putting in the place of a and b, their values, we have,

$$x = \pm\sqrt{-3\times(4)^2 \pm \sqrt{\tfrac{1312}{2} + 8\times(4)^4}} = \sqrt{-48 \pm \sqrt{656 + 2048}}.$$

$$= \sqrt{-48 \pm 52} = \sqrt{4} = 2.$$

Half the difference of the numbers is, therefore, 2, and since half the sum is 4, the greater number is 6, and the lesser 2.

6. Find two numbers, such that the sum of their squares being subtracted from three times their product, 11 will remain; and the difference of their squares being subtracted from twice their product, the remainder will be 14.

Let $x =$ the greater number, and $y =$ the less;

then, $3xy - (x^2 + y^2) = 11$;

and, $2xy - (x^2 - y^2) = 14$;

or performing the subtraction indicated,

$$\begin{cases} 3xy - x^2 - y^2 = 11 \\ 2xy - x^2 + y^2 = 14. \end{cases}$$

Making $xz = y$, and substituting it in the place of y, the equations become

$$\begin{cases} 3x^2z - x^2 - x^2z^2 = 11 \\ 2x^2z - x^2 + x^2z^2 = 14. \end{cases}$$

Dividing the first of these equations by the second, and casting out the factor x^2 from both numerator and denominator of the first member of the resulting equation, we have,

$$\frac{3z - 1 - z^2}{2z - 1 + z^2} = \frac{11}{14}.$$

Clearing of fractions,

$$42z - 14 - 14z^2 = 22z - 11 + 11z^2.$$

Transposing and reducing,

$$25z^2 - 20z = -3,$$

and $$z^2 - \frac{4}{5}z = -\frac{3}{25}.$$

Completing the square,

$$z^2 - \frac{4}{5}z + \left(\frac{2}{5}\right)^2 = -\frac{3}{25} + \frac{4}{25} = \frac{1}{25}.$$

Extracting the root, $z - \frac{2}{5} = \pm \frac{1}{5}$;

whence, $$z = \frac{2}{5} \pm \frac{1}{5} = \frac{3}{5}.$$

Substituting this value of z in the equation

$$3x^2z - x^2 - x^2z^2 = 11,$$

we have, $$\frac{9x^2}{5} - x^2 - \frac{9x^2}{25} = 11,$$

$$45x^3 - 25x^2 - 9x^2 = 275,$$

$$11x^2 = 275,$$

$$x^2 = 25,$$

$$x = 5;$$

but $xz = y$, therefore, $y = 5 \times \frac{3}{5} = \frac{15}{5} = 3.$

The numbers are, therefore, 5 and 3.

7. A person bought cloth for £33. 15s. which he sold again at £2. 8s. per piece, and gained by the bargain as much as one piece cost him. Required the number of pieces?
Ans. 15.

8. A and B set off at the same time, to a place at the distance of 150 miles. A travels 3 miles an hour faster than B, and arrives at his journey's end 8 hours and 20 minutes before him. At what rate did each person travel per hour?
Ans. A 9 miles and B 6.

9. There are two numbers whose product is 120. If 2 be added to the lesser, and 3 subtracted from the greater, the product of the sum and remainder will also be 120. What are the numbers? Ans. 15 and 8.

10. What number is that, which being divided by the product of its digits, the quotient is 2, and if 27 be added to the number itself, the digits will be inverted? Ans. 36.

11. The sum of two numbers multiplied by the lesser is equal to 40, and their difference multiplied by the greater is 12; what are the numbers? Ans. 4 and 6.

12. Divide the number 13 into three such parts, that their squares may have equal differences, and the sum of those squares may be 75. Ans. 7, 5, and 1.

13. The sum of two numbers is 9, and the sum of their cubes 243. Required the numbers. Ans. 3 and 6.

14. The sum of two numbers is 10, and the sum of thei 4th powers 1552. Required the numbers. Ans. 4 and 6.

15. The sum of two numbers is 7, and the su
5th powers 3157. Required the numbers. Ans.

16. Find two numbers, such that their sum, their product, and the difference of their squares, may all be equal to one another. Ans. $\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$, and $\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$.

17. Divide the number 100 into two such parts, that the sum of their square roots may be 14. Ans. 64 and 36.

18. Divide the number 40 into two such parts, that twice their product shall be equal to the cube of the less number. Ans. 8 and 32.

19. A and B hired a pasture, into which A put 4 horses, and B as many as cost him 18 shillings a week. Afterwards B put in two additional horses, and found that he must pay 20 shillings a week; at what rate was the pasture hired?

Ans. B had 6 horses in the pasture at first, and the price of the whole pasture was 50 shillings per week.

20. A mercer bought a piece of silk for £16. 4s., and the number of shillings he paid per yard, was to the number of yards as 4 to 9. How many yards did he buy, and what was the price per yard?

Ans. 27 yards at 12 shillings per yard.

21. There are two numbers in the proportion of 4 to 5, and the difference of their squares is 81. What are the numbers? Ans. 12 and 15.

22. A person bought for a dollar, as many pounds of sugar as were equal to half the number of dollars he laid out. In selling the sugar, he received for every 100lbs. as many dollars as the whole had cost him, and he received on the whole $20,48. How many dollars did he lay out, and what did he give for a pound?

Ans. He laid out $16, and gave $12\frac{1}{2}$ cents per pound.

23. What two numbers are those, whose difference multiplied by the greater, produces 40, and by the less, 15? Ans. 8 and 3.

24. What two numbers are those, which being both multiplied by 27, the first product is a square, and the second, the root of that square; but being both multiplied by 3, the first product is a cube, and the second the root of that cube? Ans. 243 and 3.

25. Find two numbers, such that the less may be to the greater as the greater is to 12, and the sum of their squares 45. Ans. 3 and 6.

26. What two numbers are those whose product is 20, and the difference of their cubes 61? Ans. 4 and 5.

27. What number is that, which being divided by the product of its two digits, the quotient is $5\frac{1}{3}$; and 9 being subtracted from the number itself, the digits will be inverted? Ans. 32.

28. Divide 20 into three such parts, that the continual product of these parts may be 270; and that the difference of the first and second may be 2 less than the difference of the second and third. Ans. 5, 6, and 9.

29. Divide the number 11 into two such parts, that the product of their squares may be 784. Ans. 4 and 7.

30. Find two numbers, such that the sum of their squares may be 89, and their sum multiplied by the greater may produce 104. Ans. 5 and 8.

CHAPTER V.

PROGRESSIONS AND LOGARITHMS.

ARITHMETICAL PROGRESSION.

(78.) An *arithmetical progression*, or *progression by differences*, is a series of numbers, in which the terms continually increase or decrease by a constant number, which is called the *ratio* or *common difference* of the progression, (*Arith.* 60).

Thus, 1, 3, 5, 7, 9, 11, &c. is an increasing progression; and, 11, 9, 7, 5, 3, 1, is a decreasing progression.

The ratio or common difference in these progressions is 2.

In order to investigate the properties of arithmetical progressions, we will take the general progression,

$$a,\ b,\ c,\ d,\ e,\ f\ .\ .\ .\ .\ l\,;$$

the first term of which is a, the last term l, the *common difference* q, and the number of terms n.

(79.) From the nature of an arithmetical progression, we have,

$$b=a+q,\ c=b+q,\ d=c+q,\ e=d+q,\ \&c.;$$

or, by putting the value of b in the equation $c=b+q$, and the value of c as found in this, in the next, and so on, we shall have,

$$b=a+q,\ c=a+2q,\ d=a+3q,\ e=a+4q,\ \&c.;$$

and writing these values in the general progression, it becomes,

$$a,\ a+q,\ a+2q,\ a+3q,\ a+4q,\ \&c.,$$

from which it will be seen, that any term of the progression may be found by adding to a as many times the common difference as there are terms before this term, or as many times the common difference as there are terms in the progression less ONE. Hence, to find the last term of a progression, we have the formula,

$$l=a+(n-1)q.$$

If the progression be a decreasing one, we have only to make q negative, and the formula will become,

$$l=a+(n-1)\times-q,$$

or, $$l=a-(n-1)q$$

From these formulas is derived the principle laid down in arithmetic (62), namely :—

In an increasing arithmetical progression, the last term is equal to the first term, *plus* the common difference multiplied by the number of terms less 1; and in a decreasing progression the last term is equal to the first, *minus* the common difference multiplied by the number of terms less 1.

(80.) Let x denote any term of a progression which has p terms before it, and y a term which has p terms after it, then, by the last formula, we shall have,

$$x=a+p\times q;$$

and, $$y=l-p\times q.$$

And adding these equations together, we obtain the equation,

$$x+y=a+l,$$

which shows *that the sum of any two terms, equally distant from the extremes, is equal to the sum of the extremes* (Arith. 61).

(81.) If x be the middle term of a progression, having an odd number of terms, then

$$x = a + p \times q,$$

and, $$x = l - p \times q;$$

and adding, we obtain $2x = a + l$.
In this case, therefore, *the sum of the extremes is equal to double the middle term* (Arith. 61).

(82.) If, from the equation $l = a + (n - 1)q$, we find the value of q, we shall have,

$$q = \frac{l - a}{n - 1};$$

or, *the common difference of an arithmetical progression, equal to the difference of the extremes divided by the number of terms less* 1.

(83.) In order to find the sum of any number of terms of an arithmetical progression, let $S =$ the sum, and we shall have, $S = a + b + c + d + e + f \dots + i + k + l$,
and we shall also have,

$$S = l + k + i, \dots\dots\dots + c + b + a.$$

Adding these equations together we obtain,

$$2S = (a+l)+(b+k)+(c+i) \dots\dots +(c+i)+(b+k)+(a+l).$$

Since the parts $(a + b)$ $(b + k)$ &c. are equal to each other, each being the sum of two terms equally distant from the extremes, and therefore equal to $(a + l)$, the sum of the extremes (80), we shall have in a progression of n terms,

$$2S = (a + l)n;$$

and dividing by 2, $$S = \frac{(a + l)n}{2}.$$

Hence we conclude, *that the sum of the terms of an arithmetical progression is equal to the sum of the extremes, multiplied by the number of terms, and divided by* 2 (Arith. 64).

(84.) From the formulas

$$l = a + (n - 1)q,$$

$$S = \frac{(a + l)n}{2},$$

other expressions may be derived; we will give some of the most useful.

(85.) From the first we have, by considering all the quantities which enter into it, except n, known,

$$n = \frac{l - a}{q} + 1;$$

that is, *the number of terms equal to the difference of the extremes divided by the common difference*, plus 1.

(86.) From the same equation, we obtain, by transposing a and l, and changing the signs,

$$a = l - (n - 1)q;$$

or, *the first term equal to the last term*, minus *the common difference multiplied by the number of terms less* 1.

(87.) The equation $S = \frac{(a + l)n}{2}$ may be changed so as to express each of the quantities S, a, l, and n, in terms of the others. Thus, by making n, l, and a, successively, to stand alone in one member of the equation, we obtain the three following equations:—

$$n = \frac{2S}{a + l}. \quad l = \frac{2S}{n} - a. \quad a = \frac{2S}{n} - l.$$

From the first of these we conclude, *that if the sum of a progression and the extremes are known, we shall find the number of terms by dividing twice the sum by the sum of the extremes.*

From the second and third, *that the sum, the number of terms, and one extreme being given, the other extreme may be found by dividing twice the sum by the number of terms, and subtracting the given extreme from the quotient.*

(88.) If in the equation $S = \frac{(a - l)n}{2}$, we substitute in the place of l, its value, as found in the equation $l = a + (n + 1)q$,

we shall have, $$S = \frac{[a + a + (n - 1)q]n}{2};$$

or, $$S = \frac{[2a + (n - 1)q]n}{2},$$

a formula, from which we can find the sum of a progression, when the first term, the number of terms, and the common ence, are known.

From this formula, we may deduce three others, expressing the values of n, q, and a, respectively, the others being given.

Thus, clearing the equation of fractions, we have,

$$2S = [2a + (n - 1)q]n.$$

Performing the multiplication indicated,

$$2S = 2an + qn^2 - qn.$$

Transposing, $qn^2 + 2an - qn = 2S$;

or, $n^2 + \frac{2an}{q} - \frac{qn}{q} = \frac{2S}{q}$.

Completing the square and reducing, we find,

$$n = -\frac{2a - q}{2q} \pm \sqrt{\frac{2S}{q} + \left(\frac{2a - q}{2q}\right)^2}.$$

(89.) From the formula $S = \frac{[2a + (n - 1)q]n}{2}$.

We have, by clearing of fractions, and dividing by n,

$$\frac{2S}{n} = 2a + (n - 1)q.$$

Transposing and reducing, $(n - 1)q = \frac{2S - 2an}{n}$;

$$q = \frac{2S - 2an}{n(n - 1)}.$$

From the same formula, we may also find the value of a, which is, $a = \frac{S}{n} - \frac{(n-1)q}{2}$.

(90.) These last formulas are too complicated for general use, and cannot easily be expressed in the form of a rule. They will serve to exercise the learner in the application of algebraic formulas to the solution of numerical problems.

We will give a single example, making use of the most complicated of the preceding formulas.

It is proposed to determine the number of terms of an arithmetical progression, the first term of which is 4, the common difference 3, and the sum of all the terms 116.

To do this, it is only necessary to substitute 4, 3, and 116, in the place of a, q, and S, respectively, in the expression

$$n = -\frac{2a - q}{2q} \pm \sqrt{\frac{2S}{q} + \left(\frac{2a - q}{2q}\right)^2}.$$

Making these substitutions, we have,

$$n = -\frac{8-3}{6} \pm \sqrt{\frac{232}{3} + \left(\frac{8-3}{6}\right)^2},$$

$$\text{or, } n = -\frac{5}{6} \pm \sqrt{\frac{232}{3} + \left(\frac{5}{6}\right)^2} = -\frac{5}{6} \pm \sqrt{\frac{232}{3} + \frac{25}{36}}$$

$$= -\frac{5}{6} \pm \sqrt{\frac{2809}{36}} = -\frac{5}{6} \pm \frac{53}{6} = \frac{48}{6} = 8.$$

The number of terms in this progression is therefore 8.

(91.) We will now give a number of examples, in which the preceding formulas may be applied, and, for the sake of brevity, shall, as before, designate the first term of a progression by a, the last by l, the number of terms by n, the common difference by q, and the sum of all the terms by S.

EXAMPLES.

1. In an increasing progression $a = 2$, $q = 3$, and $n = 9$. Required l and S. Ans. $l = 26$, and $S = 126$.

2. In an increasing progression $a = 3$, $n = 12$, and $l = 24$. Required q and S. Ans. $q = 1\frac{10}{11}$, and $S = 162$.

3. In an increasing progression $S = 200$, $l = 21$, and $n = 10$. Required a and q. Ans. $a = 19$, and $q = \frac{2}{9}$.

4. In an increasing progression $l = 78$, $S = 728$, and $n = 14$, to find a and q. Ans. $a = 26$ and $q = 4$.

5. In an increasing progression $a = 66$, $S = 816$, and $n = 24$. Required l and q.

By applying the formula, it will be perceived that this is a decreasing progression in which $l = 2$, and $q = -2\frac{18}{23}$.

6. What debt can be discharged in a year by paying 1 cent the first day, 3 cents the second, 5 cents the third, and so on, increasing the payment each day by 2 cents.
Ans. 1332 dollars 25 cents.

7. A footman travels the first day 20 miles, 23 the second, 26 the third, and so on, increasing the distance each day 3 miles; how may days must he travel at this rate to go 438 miles? Ans. 12 days.

8. Two persons set out from the same place, and travel in the same direction. The first goes 45 miles the first day, 43

the second, &c., travelling each day 2 miles less than the preceding. The second goes 21 miles the first day, 24 the second, &c., increasing each day 3 miles. What will be the distance between them at the end of ten days?

Ans. 15 miles.

PROBLEMS IN ARITHMETICAL PROGRESSION, PRODUCING EQUATIONS.

(92.) 1. There are three numbers in arithmetical progression, their sum is 18, and the sum of their squares 158. What are those numbers?

Let x = the first, and y = the last, then $\frac{x+y}{2}$ will be the other (81), and we shall have the equations,

$$\begin{cases} x+\frac{x+y}{2}+y = 18 \\ x^2+\left(\frac{x+y}{2}\right)^2+y^2=158. \end{cases}$$

Clearing the first equation of fractions, we obtain,

$$2x+x+y+2y=36;$$

reducing, $x+y=12.$

Putting this value of $(x+y)$ in the second equation, that equation becomes,

$$x^2+\left(\frac{12}{2}\right)^2+y^2=158;$$

or, $x^2+36+y^2=158;$

and, $x^2+y^2=158-36=122.$

From the equation $x+y=12$, we have, $y=12-x$. Squaring this value of y, and putting it in the place of y^2, in the last equation, we have,

$$x^2+144-24x+x^2=122.$$

Transposing and reducing, $x^2-12x=-11,$

$$x^2-12x+36=-11+36=25,$$

$$x-6=\pm 5,$$

$$x=11, \text{ or } 1,$$

$$y=12-11=1,$$

$$\frac{x+y}{2}=\frac{11+1}{2}=6.$$

The numbers are therefore 1, 6, and 11.

Another Solution.

Let $x =$ the second number, and $y =$ the common difference; then $x - y$, x, and $x + y$ will be the numbers (79), and we shall have the equation,

$$x - y + x + x + y = 18;$$

and reducing,
$$3x = 18,$$
$$x = 6.$$

By the second condition, the sum of their squares is 158; therefore, $(6 - y)^2 + (6)^2 + 6 + y)^2 = 158;$

$$\text{or, } 36 - 12y + y^2 + 36 + 36 + 12y + y^2 = 158.$$

Transposing and reducing, $2y^2 = 50,$
$$y^2 = 25,$$
$$y = 5,$$
$$x - y = 6 - 5 = 1,\ x = 6,\ x + y = 6 + 5 = 11.$$

1, 6, and 11, are therefore the numbers.

2. There are five numbers in arithmetical progression; the sum of these numbers is 65, and the sum of their squares 1005. What are the numbers?

Let $x =$ the first, and $y =$ the common difference, then we shall have (79),

$$x + (x + y) + (x + 2y) + (x + 3y) + (x + 4y) = 65$$
$$x^2 + (x + y)^2 + (x + 2y)^2 + (x + 3y)^2 + (x + 4y)^2 = 1005.$$

By performing the operations indicated, and reducing, these equations become $x + 2y = 13,$
$$x^2 + 4xy + 6y^2 = 201.$$

Squaring the first of these equations, and subtracting it from the second, there remains $2y^2 = 32$;
from which we find $y = 4$, and by putting this value of y in the first equation, we find $x = 5$.
The numbers are therefore 5, $5 + 4$, $5 + 8$, $5 + 12$, $5 + 16$, or, 5, 9, 13, 17, and 21.

In the same manner we might find any number of terms in an arithmetical progression, their sum, and the sum of their squares being given.

3. A number consists of three digits which are in arithmetical progression; and if the number be divided by the

sum of its digits, the quotient will be 26; but if 198 be added to it, the digits will be inverted. Required the number.

Ans. 234.

4. A person bought 7 books, the prices of which were in arithmetical progression. The price of the next above the cheapest was eight shillings; and the price of the dearest 23 shillings. What was the price of each book?

Ans. 5, 8, 11, 14, 17, 20, and 23 shillings, respectively.

5. A and B, 165 miles distant from each other, set out with a design to meet; A travels 1 mile the first day, 2 the second, 3 the third, and so on; B travels 20 miles the first day, 18 the second, 16 the third, &c. In how many days will they meet, and how far will each have travelled?

Ans. They meet in 10 days, A having travelled 55 miles, and B 110.

GEOMETRICAL PROGRESSION.

(93.) A *geometrical progression*, or a *progression by quotients*, is a series of numbers, having this property, namely, each term of the series, except the first, is the product of the one that precedes it by a *constant number*. This number is called the *ratio* of the progression. A geometrical progression is written in the following manner (Arith. 72):—

$$2 : 6 : 18 : 54 : 162 : 486.$$

In this progression the ratio is 3, each term being three times that which precedes it. When the ratio is greater than unity, the progression is called an increasing progression; and when the ratio is less than unity, it is called a decreasing progression;

$$81 : 27 : 9 : 3 : 1 : \tfrac{1}{3} : \tfrac{1}{9} : \tfrac{1}{27}$$

is a decreasing progression, the ratio of which is $\frac{1}{3}$.

In order to investigate the properties of geometrical progression, let the first term be represented by a, the last by l, the ratio by r, the sum of all the terms by S, and the number of terms by n.

(94.) Since each term of the progression is produced by multiplying the term that precedes it by the ratio, the first term being a, the second will be $a \times r$, or ar; the third, $ar \times r$, or ar^2; the fourth, ar^3, and so on. We shall have, therefore, for the general expression of a geometrical progression,

$$a : ar : ar^2 : ar^3 : ar^4 : ar^5 \;.\;.\;.\;.\;.\; ar^{n-2} : ar^{n-1}.$$

In this progression, it will be perceived, that the exponent of r in each term, is equal to the number of terms that precede it; and hence, in an equation of n terms, the last term having $n-1$ terms before it, will be ar^{n-1}; and calling this last term l, as was proposed, we have,

$$l = ar^{n-1}.$$

Hence we conclude that, *in every geometrical progression, the last term is equal to the first term, multiplied by that power of the ratio, which is equal to the number of terms less* 1 (Arith. 74).

(95.) Let x be a term of a geometrical progression which has p terms before it, we shall have (94),

$$x = ar^{p}.$$

Let y be a term of the same progression which has p terms after it; then, since the exponents of r diminish by unity, for each term, proceeding from the last towards the first, the term which has p terms after it, will be represented by ar^{n-1-p}; wherefore, $y = ar^{n-1-p}$.

Multiplying this equation, term by term, by the preceding, there results, $xy = a^2r^{n-1-p+p} = a^2r^{n-1}$.

If the progression have an odd number of terms, and x be the middle term, then $x = y$, and we shall have,

$$x^2 = a^2r^{n-1}.$$

But in any geometrical progression, a being the first term, and ar^{n-1} the last, we shall have,

$$a \times ar^{n-1}, \text{ or } a^2r^{n-1},$$

for the product of the extremes.

Hence we conclude that, *in every geometrical progression the product of the extreme terms is equal to the product of any two mean terms, equally distant from them; and also equal to the square of the middle term, when there is an odd number of terms* (Arith. 74).

(96.) Let $a : b : c : d : e$, &c. be a geometrical progression. From the definition of a geometrical progression, we shall have, $b = ar,\ c = br,\ d = cr,\ e = dr\ \ldots\ldots\ l = kr$.

Adding these equations together, we obtain,

$$b + c + d + e \ldots\ldots + l = ar + br + cr + dr + er \ldots\ldots + kr,$$

$$\text{or, } b + c + d + e \ldots\ldots + l = (a + b + c + d + e \ldots\ldots + k)r.$$

The first member of this equation contains all the terms of the progression except a; and hence if S be the sum of all the terms, it is equal to $S-a$. The second member contains all the terms of the same progression except l, mulplied by r; it is therefore equal to $(S-l)r$;
whence, $S-a=(S-l)r$; or, $S-a=rS-rl$.

Transposing, $rS-S=rl-a$; whence, $S=\frac{rl-a}{r-1}$.

From which it appears, that *the sum of all the terms of a geometrical progression, is equal to the product of the last term by the ratio,* minus *the first term, divided by the ratio diminished by unity* (Arith. 75).

(97.) The equations $l=ar^{n-1}$, and $S=\frac{rl-a}{r-1}$, express the relations which exist between the quantities a, n, l, r, and S, and furnish the means of determining any two of these quantities, the other two being known.

If in the equation $S=\frac{rl-a}{r-1}$, we substitute in the place of l its value ar^{n-1}, as found in the other, this last equation becomes

$$S=\frac{ar^n-a}{r-1}, \text{ or } S=\frac{a(r^n-1)}{r-1}.$$

By means of this formula, the sum of a progression may be determined without knowing the last term.

(98.) The equation $S=\frac{rl-a}{r-1}$ gives, by clearing of fractions and reducing,

$$rS-S=rl-a, \text{ or } rS-rl=S-a;$$

whence, $$r=\frac{S-a}{S-l}.$$

From which we learn, that *the ratio of any progression is equal to the sum,* minus *the first term, divided by the sum,* minus *the last term* (Arith. 76).

(99.) If in the equation $S=\frac{a(r^n-1)}{r-1}$, we consider r greater than unity, it is evident that S may be made greater than any quantity however large; for the quantity r^n may be made greater than any assignable quantity by increasing the value of n sufficiently; that is, by taking a sufficient number of terms in the progression.

But, if r be a fraction less than unity, there will be a limit to the value of S, which we now propose to show.

Let $r = \frac{1}{m}$, then the equation $S = \frac{a(r^n - 1)}{r - 1}$ becomes

$$S = \frac{a\left(\frac{1}{m^n} - 1\right)}{\frac{1}{m} - 1} = \frac{am\left(\frac{1}{m^n} - 1\right)}{1 - m}.$$

But since $m > 1$, and $\frac{1}{m^n} < 1$, we change the signs of the terms of the fraction $\frac{\frac{1}{m^n} - 1}{1 - m}$, so as to make both terms positive; and the equation then becomes,

$$S = \frac{am\left(1 - \frac{1}{m^n}\right)}{m - 1} = \frac{am - \frac{am}{m^n}}{m - 1} = \frac{am - \frac{a}{m^{n-1}}}{m - 1}.$$

The numerator of this fraction will become greater, as the quantity $\frac{a}{m^{n-1}}$ diminishes; and it is evident that the greater the value of n, the less will be the value of $\frac{a}{m^{n-1}}$; and hence it will appear, that the greater the number of terms, the nearer the value of S will approach to $\frac{am}{m - 1}$, from which it differs only by the quantity $\frac{\frac{a}{m^{n-1}}}{m - 1}$, or $\frac{a}{(m - 1)m^{n-1}}$. When n is very large, this quantity may be less than any assignable quantity, and in that case, $\frac{am}{m - 1}$ may be taken for the value of S, though it cannot be said to be equal to it in a rigorous sense. It is the limit towards which the value of S approaches, as the number of terms of the progression increases.

For example; let us take the progression

$$1 : \tfrac{1}{3} : \tfrac{1}{9} : \tfrac{1}{27} : \tfrac{1}{81} : \tfrac{1}{243}, \text{ \&c.}$$

Here we have $a = 1$, $r = \frac{1}{3}$, and, consequently, $m = 3$; then,*

$$\frac{am}{m-1} = \frac{3}{2};$$

and it will be seen that the greater the number of terms we take, the nearer the sum of these terms will approach to $\frac{3}{2}$.

$$\begin{array}{lcl}
1 & = 1 & = \frac{3}{2} - \frac{1}{2} \\
1+\frac{1}{3} & = \frac{4}{3} & = \frac{3}{2} - \frac{1}{6} \\
1+\frac{1}{3}+\frac{1}{9} & = \frac{13}{9} & = \frac{3}{2} - \frac{1}{18} \\
1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27} & = \frac{40}{27} & = \frac{3}{2} - \frac{1}{54} \\
1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81} & = \frac{121}{81} & = \frac{3}{2} - \frac{1}{162}
\end{array}$$

From this it is evident that when the number of terms is very large, we may neglect the small fraction by which the sum differs from $\frac{am}{m-1}$; and hence we take $\frac{am}{m-1}$ for the sum of a decreasing geometrical progression, when the number of terms is infinite. The quantity m may not always be a whole number, for if $r=\frac{2}{3}$, since $r = \frac{1}{m}$, we have $\frac{2}{3} = \frac{1}{m}$; hence, $m = \frac{3}{2}$, and $\frac{1}{m} = \frac{1}{\frac{3}{2}}$.

We therefore conclude, *that the sum of a decreasing progression continued to infinity, is equal to the product of the first term multiplied by* m, *and divided by* m — 1*;* m *being such a number, that* $\frac{1}{\mathrm{m}}$ *shall always be equal to the ratio of the progression.*

For an application of this principle, let it be required to find the sum of the progression:

$$18 : 12 : 8 : \tfrac{16}{3} : \tfrac{32}{9} : \tfrac{64}{27} : \tfrac{128}{81}, \text{ \&c.}$$

continued to infinity. The ratio of this progression is $\frac{2}{3}$; $\frac{1}{m} = \frac{2}{3}$ therefore, and $m = \frac{3}{2}$. We have, therefore,

$$\frac{am}{m-1} = \frac{18\times\frac{3}{2}}{\frac{3}{2}-1} = \frac{27}{\frac{1}{2}} = 54.$$

If from the formula $S = \frac{a(r^n-1)}{r-1}$, we find the sum of ten terms, we shall have $S = 53\frac{139}{2187}$, which differs from the former value, by a quantity less than unity.

(100.) If in the formula $S=\frac{a(r^n-1)}{r-1}$, we make r equal to unity, it would seem at first sight to present an absurdity; for if $r=1$, all the powers of r will equal 1, and the equation becomes, $S=\frac{a(1-1)}{1-1}$, or $S=a\frac{0}{0}$. But we shall see that the expression $\frac{0}{0}$, does not here express absolutely nothing; for if we perform the division indicated in the fraction $\frac{r^n-1}{r-1}$, or, $\frac{1-r^n}{1-r}$, (for the two expressions are equivalent), we shall find for a quotient,

$$1+r+r^2+r^3+r^4+\ldots r^{n-2}+r^{n-1};$$

and therefore we shall have,

$$S=a(1+r+r^2+r^3+r^4 \ldots +r^{n-2}+r^{n-1},$$

or, $S=a+ar+ar^2+ar^3+ar^4 \ldots ar^{n-2}+ar^{n-1}$;

which, when $r=1$, becomes,

$$S=a+a+a+a+a, \&c.,$$

which is evidently correct; for if the ratio be unity, all the terms will be equal, and the same as the first term, and their sum will be na, when there are n terms. It will be seen from what has been said, that the formulas which have been investigated, apply only when the ratio is greater or less than unity. Indeed, when the ratio is unity, the series, being only a repetition of the same number, cannot be strictly called a geometrical *progression.*

(101.) If, in the equation $S=\frac{a(r^n-1)}{r-1}$, we seek the value of n; S, a and r, being supposed to be known, we shall find

$$r^n=\frac{S(r-1)}{a}+1$$

an equation of a different kind from any that we have before considered, one in which the unknown quantity is an exponent. The general form of such equations is $a^x=b$. They are called exponential equations, the method of reducing which will be given hereafter.

(102.) We now proceed to give some examples to exercise the learner in the application of the formulas relating to geometrical progressions; and, for the sake of brevity, we shall still represent the first term by a, the last by l, the ratio by r, the number of terms by n, and the sum by S.

EXAMPLES.

1. In a geometrical progression $a = 5$, $n = 7$, and $r = 4$. Required the value of l.

$$l = ar^{n-1} = 5 \times 4^6 = 5 \times 4096 = 20480.$$

2. Given $a = 2$, $n = 8$, and $r = \frac{1}{4}$, to find l.

$$l = 2 \times (\tfrac{1}{4})^7 = 2 \times \tfrac{1}{16384} = \tfrac{2}{16384} = \tfrac{1}{8192}.$$

3. Given $a = 5$, $r = 3$, $l = 98415$, to find S.

$$S = \frac{rl - a}{r - 1} = \frac{(3 \times 98415) - 5}{3 - 1} = \frac{295240}{2} = 147620.$$

4. Given $a = 1$, $n = 8$, and $r = 5$, to find S.

$$S = \frac{a(r^n - 1)}{r - 1} = \frac{390625 - 1}{4} = 97656.$$

5. Given $a = 64$, $n = 5$, $r = \frac{1}{8}$, to find S,

$$S = \frac{64(\frac{1}{32768} - 1)}{\frac{1}{8} - 1} = \frac{\frac{64}{32768} - 64}{\frac{1}{8} - 1} = \frac{\frac{1}{512} - 64}{\frac{1}{8} - 1}$$

$$= \frac{-\frac{32767}{512}}{-\frac{7}{8}} = -\tfrac{32767}{512} \times -\tfrac{8}{7} = \tfrac{32767}{448} = 73\tfrac{63}{448}.$$

6. Given $a = 64$, $r = \frac{1}{4}$, and $n = 12$, to find l and S.
Ans. $l = \frac{1}{65536}$, and $S = 85 \times \frac{65535}{196608}$.

7. Given $l = 45056$, $S = 90101$, and $a = 11$, to find r.
Ans. $r = 2$.

8. Given $S = 49205$, $r = 3$, $n = 9$, to find a.
Ans. $a = 5$.

9. Given $l = \frac{1}{128}$, $S = 7\frac{127}{128}$, and $2 = \frac{1}{2}$, to find a.
Ans. $a = 4$.

10. Find the sum of the progression $5 : \frac{5}{3} : \frac{5}{9} : \frac{5}{27}$. &c. continued to infinity. Ans. $7\frac{1}{2}$.

11. What debt can be discharged in one year by paying 1 dollar the first month, 2 the second, 3 the third, and so on, doubling the payment every month. Ans. 4095 dollars.

12. Two couriers, A and B, set out to meet each other. A travels 6 miles the first hour, 8 the second, 10 the third, and so on, increasing his rate of travelling 2 miles, every hour. goes 3 miles the first hour, 4½ the second, 6¾ the 3d, travell

each hour, $1\frac{1}{2}$ times as far as the preceding hour. They meet after 6 hours. Required the distance between the places from which they set out. Ans. $128\frac{11}{32}$ miles.

PROBLEMS IN GEOMETRICAL PROGRESSION, PRODUCING EQUATIONS.

(103.) 1. The sum of three numbers in geometrical progression is 26, and the sum of their squares 364. What are the numbers?

Let the numbers be x, y, and z; then we shall have,

$$x+y+z=26,$$
$$x^2+y^2+z^2=364,$$

and since the product of the extremes is equal to the square of the middle term (95), $xz=y^2$,

If in the first of these equations, we transpose y into the second member, and then square both members, we shall have, $x^2+2xz+z^2=676-52y+y^2$.

Putting for xz its value, y^2, as found in the third equation, this equation becomes,

$$x^2+2y^2+z^2=676-52y+y^2;$$

or, $x^2+y^2+z^2=676-52y$.

But from the second equation, we have,

$$x^2+y^2+z^2=364;$$

therefore, $364=676-52y$;

transposing, $52y=312$; whence, $y=6$.

Substituting this value of y in the third equation, we have,

$$xz=36, \text{ or } x=\frac{36}{z}.$$

Putting this value of x in the first equation, it becomes,

$$\frac{36}{z}+z+6=26,$$
$$36+z^2+6z=26z,$$
$$z^2-20z=-36,$$
$$z-10=\pm\sqrt{-36+100}=\sqrt{64}=8,$$
$$z=10\pm8=18, \text{ or } 2,$$
$$x=\frac{36}{z}=\frac{36}{18}=2.$$

The numbers are therefore 2, 6, and 18.

2. Divide the number 210 into three parts, so that the last shall exceed the first by 90, and the parts be in geometrical progression.

Let $x =$ the first part, and $y =$ the ratio of the progression, then the parts will be x, xy, and xy^2, and we shall have,

$$x + xy + xy^2 = 210,$$

and,

$$xy^2 = x + 90.$$

By putting $x + 90$ in the place of xy^2, in the first equation, it becomes,

$$2x + xy + 90 = 210,$$

or,

$$2x + xy = 120;$$

from which we obtain, $x = \frac{120}{y+2}$.

From the equation, $xy^2 = x + 90$, we find,

$$x = \frac{90}{y^2 - 1}.$$

Putting these values of x equal, we have,

$$\frac{120}{y+2} = \frac{90}{y^2 - 1}, \text{ or } \frac{4}{y+2} = \frac{3}{y^2 - 1};$$

from which we obtain, by clearing of fractions and reducing,

$$4y^2 - 3y = 10,$$

$$y^2 - \frac{3}{4} = \frac{10}{4},$$

$$y - \frac{3}{8} = \pm\sqrt{\frac{10}{4} + \frac{9}{64}} = \pm\frac{13}{8},$$

$$y = \frac{3}{8} \pm \frac{13}{8} = 2.$$

But $x = \frac{120}{y+2} = 30$, $xy = 60$, $xy^2 = 120$.

3. The sum of four numbers in geometrical progression, is 15, and the sum of their squares 85. Required the numbers.

Let $x =$ the second term, and $y =$ the third term of the progression, then will $\frac{x^2}{y}$, and $\frac{y^2}{x}$ be the first and fourth, since we shall have $\frac{x^2}{y} \times \frac{y^2}{x} = xy$ (95).

Putting, for the sake of brevity, $15 = a$, and $85 = b$, we shall have the equations

$$\frac{x^2}{y} + x + y + \frac{y^2}{x} = a,$$

$$\frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} = b.$$

If in the first equation, we call the sum of the middle terms s, we shall have, in the first place,

$$x + y = s, \text{ and } \frac{x^2}{y} + \frac{y^2}{x} = a - s.$$

Squaring these equations, they become,

$$x^2 + 2xy + y^2 = s^2,$$

$$\frac{x^4}{y^2} + 2xy + \frac{y^4}{x^2} = (a - s)^2;$$

adding them together, and transposing $4xy$, we have,

$$\frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} = s^2 + (a - s)^2 - 4xy;$$

whence, $s^2 + (a - s)^2 - 4xy = b.$

In order to eliminate xy, take the two equations,

$$x + y = s, \text{ and } \frac{x^2}{y} + \frac{y^2}{x} = a - s.$$

Cubing the first, it becomes,

$$x^3 + 3x^2y + 3xy^2 + y^3 = s^3;$$

transposing, $x^3 + y^3 = s^3 - (3x^2y + 3xy^2)$

$= s^3 - 3xy(x + y) = s^3 - 3xys.$

Clearing the second equation of fractions, we have,

$$x^3 + y^3 = (a - s)xy;$$

wherefore, $(a - s)xy = s^3 - 3xys,$

or, $axy - sxy = s^3 - 3sxy;$

transposing, $2sxy + axy = s^3,$

and, $xy = \dfrac{s^3}{2s + a}.$

Substituting this value of xy in the equation

$$s^2 + (a - s)^2 - 4xy = b,$$

we have, $s^2 + (a - s)^2 - \dfrac{s^3}{2s + a} = b.$

Clearing this equation of fractions, and reducing, it becomes,

$$s^2+\frac{b}{a}s=\frac{a^2-b}{2};$$

from which we obtain,

$$s=-\frac{b}{2a}\pm\sqrt{\frac{a^2-b}{2}+\frac{b^2}{4a^2}}.$$

Giving to a and b their numerical values, we have,

$$s=-\frac{85}{30}\pm\sqrt{\frac{225-85}{2}+\frac{7225}{900}}=-\frac{85}{30}\pm\frac{265}{30}=6.$$

Having found the value of s, we substitute it in the equation, $xy=\frac{s^3}{2s+a}$; from which we find

$$xy=\frac{216}{12+15}=\frac{216}{29}=8.$$

We then have the two equations $xy=8$, and $x+y=6$, to find the values of x and y.

The first gives $x=\frac{8}{y}$; which value of x being substituted in the second, we have,

$$\frac{8}{y}+y=6,$$
$$8+y^2=6y,$$
$$y^2-6y=-8,$$
$$y-3=\pm\sqrt{-8+9}=\pm 1,$$
$$y=4, \text{ or } -2.$$

$$x=\frac{8}{y}=\frac{8}{4}=2,\ \frac{x^2}{y}=\frac{4}{4}=1, \text{ and, } \frac{y^2}{x}=\frac{16}{2}=8.$$

The numbers are therefore 1, 2, 4, and 8. If we had taken the negative value of y, we should have had,

$$x=4,\ \frac{x^2}{y}=\frac{16}{2}=8, \text{ and } \frac{y^2}{x}=\frac{4}{4}=1;$$

so that the result would have been the same, except that the progression $\frac{x^2}{y}+x+y+\frac{y^2}{x}$, would have been a decreasing progression.

4. The sum of four numbers in geometrical progression is 30; and the last term divided by the sum of the mean terms is $\frac{4}{3}$. What are the numbers? Ans. 2, 4, 8, and 16.

5. There are three numbers in geometrical progression, whose product is 64, and the sum of their cubes 584. What are the numbers? Ans. 2, 4, and 8.

6. The sum of three numbers in geometrical progression is 13; and the sum of the extremes being multiplied by the mean term, the product is 30. What are the numbers?
Ans. 1, 3, and 9.

7. 120 dollars are divided between four persons, in such a way, that their shares may be in arithmetical progression; but if the second and third had received 12 dollars less each, and the fourth 24 dollars more, the shares would have been in geometrical progression. Required each share.
Ans. Their shares were 3, 21, 39, and 57, respectively.

8. There are three numbers in geometrical progression, whose sum is 31, and the sum of the first and last is 26. What are the numbers? Ans. 1, 5, and 25.

9. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth 60. What are the numbers? Ans. 5, 10, 20, 40.

10. The sum of four numbers in geometrical progression is equal to the common ratio $+1$, and the first term is $\frac{1}{17}$. What are the numbers? Ans. $\frac{1}{17}$, $\frac{4}{17}$, $\frac{16}{17}$, $\frac{64}{17}$.

EXPONENTIAL EQUATIONS.

(104.) An *exponential equation* is one in which the unknown quantity is an *exponent*; as $a^x = b$, $3^x = 243$.

When the exponent is a whole number, such an equation presents no difficulty; for example, if we would find the value of x in the equation $3^x = 243$, we are only to raise 3 to such a power as will equal 243; and the exponent of this power is the value of x.

We find $3^5 = 243$, and therefore $x = 5$. If, however, we have $3^x = 54$, we can only obtain the value of x by approximation, and the approximate value will be a fraction.

We now proceed to solve the equation

$$3^x = 54.$$

If we raise 3 to the third power, we have,

$$3^3 = 27:$$

x is therefore greater than 3; and it is less than 4, for

$3^4 = 81$. We must therefore take for the value of x, $3 +$ a fraction. Let that fraction be $\frac{1}{x'}$, in which x is greater than unity. We then have,

$$3^{3+\frac{1}{x'}} = 54, \text{ or } 3^3 \times 3^{\frac{1}{x'}} = 54,$$

or, $$27 \times 3^{\frac{1}{x'}} = 54.$$

Dividing by 27, $$3^{\frac{1}{x'}} = \tfrac{54}{27} = 2.$$

Raising both members of this equation to the x'th power, it becomes, $$3 = 2^{x'}.$$

Here the value of x' is less than 2; for $2^2 = 4$; and it is greater than one, since $2^1 = 2$; x' therefore equals $1 +$ a fraction. Putting $x' = 1 + \frac{1}{x''}$, we have,

$$2^{1+\frac{1}{x''}} = 3, \text{ or, } 2 \times 2^{\frac{1}{x''}} = 3.$$

Dividing by 2, $$2^{\frac{1}{x''}} = \tfrac{3}{2};$$

and raising both members to the x'' power, we have,

$$2 = (\tfrac{3}{2})^{x''}.$$

But $(\frac{3}{2})^2 = \frac{9}{4}$, is greater than 2, and $\frac{3}{2}$ is less; wherefore, $x'' = 1 +$ a fraction. Making $x'' = 1 + \frac{1}{x'''}$, we have,

$$(\tfrac{3}{2})^{1+\frac{1}{x'''}} = 2, \text{ or } \tfrac{3}{2} \times (\tfrac{3}{2})^{\frac{1}{x'''}} = 2;$$

dividing by $\frac{3}{2}$, $$\tfrac{3}{2}^{\frac{1}{x'''}} = 2 \div \tfrac{3}{2} = \tfrac{4}{3},$$

from which we obtain, $\frac{3}{2} = (\frac{4}{3})^{x'''}$.

But $(\frac{4}{3})^2 = \frac{16}{9} > \frac{3}{2}$, and $\frac{4}{3} < \frac{3}{2}$; we may therefore put $x''' = 1 + \frac{1}{x''''}$, and then we have,

$$(\tfrac{4}{3})^{1+\frac{1}{x''''}} = \tfrac{3}{2}, \text{ or } \tfrac{4}{3} \times (\tfrac{4}{3})^{\frac{1}{x''''}} = \tfrac{3}{2}.$$

Dividing by $\frac{4}{3}$, $$(\tfrac{4}{3})^{\frac{1}{x''''}} = \tfrac{9}{8},$$

from which we obtain, $\frac{4}{3} = (\frac{9}{8})^{x''''}$.

We shall here find x'''' greater than 2, and less than 3; for $\frac{81}{64} < \frac{4}{3}$, and $\frac{729}{512} > \frac{4}{3}$. Putting $x'''' = 2 + \frac{1}{x'''''}$, we have,

$$(\tfrac{9}{8})^{2+\frac{1}{x'''''}} = \tfrac{4}{3}, \text{ or } \tfrac{81}{64} \times (\tfrac{9}{8})^{\frac{1}{x'''''}} = \tfrac{4}{3}.$$

Dividing by $\frac{81}{64}$, $\quad (\frac{9}{8})^{\frac{1}{x'''''}} = \frac{256}{243}$.

By operating upon this last equation, as upon the others, we shall find that $x''''' = 2$ nearly. Taking 2 for the value of x''''', we have,

$$x = 3 + \frac{1}{x'},\ x' = 1 + \frac{1}{x''},\ x'' = 1 + \frac{1}{x'''},\ x''' = 1 + \frac{1}{x''''},$$

$$x'''' = 2 + \frac{1}{x'''''},\ \text{and}\ x''''' = 2.$$

By substituting for x', x'', x''', &c., their values, we have, for the value of x, the continued fraction,

$$x = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2.}}}}}$$

Reducing this continued fraction to a simple one by (Arith. 42), we have, $x = 3 + \frac{12}{19} = \frac{69}{19} = 3{,}631579$.

We have, therefore, $3^x = 3^{3{,}631579}$; and putting this value of x in the original equation, it becomes,

$$3^{3{,}631579} = 54.$$

For another example, take the equation $(10)^x = 2$.

Here it is obvious, that x must be less than unity, or between 0 and unity; for $(10)^0 = 1$, and $(10)^1 = 10$. We therefore put $x = \frac{1}{z}$, and the equation becomes,

$$(10)^{\frac{1}{z}} = 2;$$

raising both members to the power z, we have,

$$2^z = 10.$$

Here z is greater than 3, and less than 4, since $2^3 = 8 < 10$, and $2^4 = 16 > 10$. Making $z = 3 + \frac{1}{z'}$, we have,

$$2^3 \times 2^{\frac{1}{z'}} = 10, \text{ or } 8 \times 2^{\frac{1}{z'}} = 10.$$

Dividing by 8, $\quad 2^{\frac{1}{z'}} = \frac{10}{8} = \frac{5}{4}$.

Raising both members to the power z', we obtain,

$$2 = (\tfrac{5}{4})^{z'}.$$

z' is here between 3 and 4, and we may make it $= 3 + \frac{1}{z''}$;

then, $\quad (\frac{5}{4})^{3+\frac{1}{z''}} = 2$, or $\frac{125}{64} \times (\frac{5}{4})^{\frac{1}{z''}} = 2$;

and dividing by $\frac{125}{64}$, $\left(\frac{5}{4}\right)^{\frac{1}{z''}} = \frac{128}{125}$;

whence, $\frac{5}{4} = \left(\frac{128}{125}\right)^{z''}$.

By continuing the operation, we should find z'', between 9 and 10. Calling z'' 9, and neglecting the succeeding fraction, $\frac{1}{z'''}$, we have the following values:—

$$x = \frac{1}{z}, \quad z = 3 + \frac{1}{z'}, \quad z' = 3 + \frac{1}{z''}, \text{ and } z'' = 9.$$

These values form a continued fraction, and we shall have,

$$x = \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{9}}}$$

The value of this fraction is $\frac{28}{93} = 0{,}30107$;

hence, $(10)^{0{,}30107} = 2$.

EXAMPLES.

1. Find the value of x in the equation $4^x = 8$.
Ans. $x = 1{,}5$.

2. Find the value of x in the equation $8^x = 3$.
Ans. 0,38.

3. Find the value of x in the equation $8^x = 128$.
Ans. 2,3333, &c.

LOGARITHMS.

(105.) If in the equation $a^x = b$, we give b different values, the value of a remaining the same, the value of x will vary as b varies; and reciprocally, by giving x different values, the value of b will vary as x varies; from which it follows, that a being considered greater or less than unity, we may give it such an exponent as will make it equal to any number whatever.

Let us suppose $a = b$; we shall also have $a^2 = b^2$, $a^3 = b^3$, &c. We have $a^0 = 1$ (39); hence for all numbers between 1 and b, the exponent of a will be a fraction less than unity; for all between b and b^2, it will be 1 + a fraction; for all be tween b^2 and b^3, $2+$ a fraction, &c. Moreover, since $a^{-1}=\frac{1}{a}$, $a^{-2}=\frac{1}{a^2}$, &c. (39), we have, $a^{-1}=\frac{1}{b}$, $a^{-2}=\frac{1}{b^2}$, $a^{-3}=\frac{1}{b^3}$, it will follow, that for all fractional numbers between 1 and $\frac{1}{b}$, the exponent of a will be a negative fraction, less than unity; for all between $\frac{1}{b}$ and $\frac{1}{b^2}$, $1+$ a fraction; for all between $\frac{1}{b^2}$ and $\frac{1}{b^3}$, $2+$ a fraction, &c., these last exponents being all negative.

If we take a less than unity, the same principles will apply, with this difference only; the negative powers of a will correspond to numbers greater than unity, and the positive powers, to numbers less than unity. For let $a=\frac{1}{a'}$, a' being greater than unity, we shall have,

$$a^{-1}=\left(\frac{1}{a'}\right)^{-1}=\frac{1}{\frac{1}{a'}}=a',\ a^{-2}=\left(\frac{1}{a'}\right)^{-2}=\frac{1}{\frac{1}{a'^2}}=a'^2,\ \&c.;$$

$$\text{and,}\quad a=\frac{1}{a'}=a'^{-1},\ a^2=\frac{1}{a'^2}=a'^{-2},\ \&c.$$

(106.) The power to which a constant number, a, must be raised, to make it equal to a number, is called the logarithm of that number. Thus, if we have $a^x=y$, and $a^{x'}=y'$, x is the logarithm of y, and x' the logarithm of y'. The logarithm of a number is denoted by writing l. before that number: thus, l. y signifies the logarithm of y.

If we take a constant number, a, and calculate by Art. (104), the values of x, which will make $a^x=1, 2, 3, 4, 5, 6, 7, 8$, &c., and arrange in a table these values of x, opposite the given numbers, we shall have a table of logarithms.

The constant number, a, is called the *base* of the table, and in the common tables is *ten*; a number on some accounts most convenient, though tables may be calculated to any other base.

Taking 10 for the base of the tables, we have,

$$(10)^0 = 1,\ (10)^1 = 10,\ (10)^2 = 100,\ (10)^3 = 1000,$$
$$(10)^4 = 10000\,;$$

from which we see that

$$l.1 = 0,\ l.10 = 1,\ l.100 = 2,\ l.1000 = 3.$$

The logarithms of numbers between 1 and 10 are expressed by decimal fractions; of those between 10 and 100, by unity, *plus* a fraction; of those between 100 and 1000, by 2 *plus* a fraction, &c.

(107.) The logarithm of any number, N, may be found by reducing the equation $(10)^x = N$, N representing the given number. Thus, in a former article, we found the value of x in the equation $(10)^x = 2$, to be 0,30107, which is the logarithm of 2. This fraction is a little too large, and by continuing the operation further, we should have obtained, 0,3010300, which is exact for seven places of decimals. In like manner, by reducing the equation $(10)^x = 3$, we find x', or the logarithm of 3, equal to 0,4771213. In the same way we might find the logarithm of any number, but as the process is exceedingly tedious, mathematicians have devised more expeditious methods, which cannot here be made known.

(108). It is only necessary to calculate directly the logarithms of prime numbers, as the logarithms of all others may be derived from these. Thus, if we would find the logarithm of 4, we have only to double that of 2; for, taking the equation $(10)^x = 2$, and squaring both members, we have, $(10)^{2x} = 4$; or taking the same equation, and cubing both members, we have $(10)^{3x} = 8$; which shows, that twice the log. of $2 = l.4$, and three times the log. of $2 = l.8$.

Again; if we would find the log. of any number, as 30, we have only to add together the logarithms of its prime factors, $30 = 3 \times 5 \times 2$, and we have,

$$(10)^x = 3,\ (10)^{x'} = 5,\ \text{and } (10)^{x''} = 2.$$

Now, if we multiply these three equations together, term by term, observing the rule for the exponents, in the first members, we shall have,

$$(10)^{x+x'+x''} = 30,\ \text{or } l.3 + l.5 + l.2 = l.30.$$

Tables of logarithms have been calculated for numbers extending from 1 to 100,000. For a description of their ment, and the manner of using them, the student to the introductions usually accompanying these tables, in which all necessary information is given.

(109.) We proceed now to show how numerical calculations are performed by means of logarithms.

Multiplication by Logarithms.

Let a be the base of a system of logarithms, and let y and y' be two numbers, the product of which is required; we shall have (106), $a^x = y,\ a^{x'} = y'$;

and multiplying these equations together, we have,

$$a^{x+x'} = yy', \text{ or } \mathrm{l}.y + \mathrm{l}.y' = \mathrm{l}.yy';$$

which shows, *that the sum of the logarithms of two numbers is equal to the logarithm of their product.*

Hence, to multiply by logarithms, we are to add together the logarithms of two factors, and find in the table the number corresponding to their sum.

Division by Logarithms.

(110.) Suppose we have, as before, $a^x = y,\ a^{x'} = y'$. Dividing the first of these equations by the second, and observing that this division is effected in the first member, by subtracting the exponent x' from x, we have,

$$a^{x-x'} = \frac{y}{y'}, \text{ or } \mathrm{l}.y - \mathrm{l}.y' = \mathrm{l}.\frac{y}{y'};$$

from which we see, *that the difference of the logarithms of two numbers, is equal to the logarithm of their quotient.*

Hence, division is performed in logarithms by subtracting the logarithm of the divisor from the logarithm of the dividend, and finding the number, in the tables, which corresponds to the difference.

Involution by Logarithms.

(111.) If we take the equation $a^x = y$, and raise both members to the 2d, 3d, 4th, and nth powers, we shall have (38),

$$a^{2x} = y^2,\ a^{3x} = y^3,\ a^{4x} = y^4,\ a^{nx} = y^n;$$

from which we see, that

$$2\,\mathrm{l}.y = \mathrm{l}.y^2,\ 3\,\mathrm{l}.y = \mathrm{l}.y^3,\ 4\,\mathrm{l}.y = \mathrm{l}.y^4,\ n\,\mathrm{l}.y = \mathrm{l}.y^n;$$

and hence, to involve a number by logarithms, we *multiply the logarithm of the number, by the number denoting the* *degree of the power, and find, in the tables, the number corresponding to the product.*

Evolution by Logarithms.

(112.) Let $a^x = y$: then if we extract the 2d, 3d, 4th, and nth roots of both members of this equation, we shall have, (45,)

$$a^{\frac{x}{2}} = \sqrt{y},\ a^{\frac{x}{3}} = \sqrt[3]{y},\ a^{\frac{x}{4}} = \sqrt[4]{y},\ a^{\frac{x}{n}} = \sqrt[n]{y};$$

or, $\frac{1}{2}\,\text{l}.\,y = \text{l}.\sqrt{y},\ \frac{1}{3}\,\text{l}.y = \text{l}.\sqrt[3]{y},\ \frac{1}{4}\,\text{l}.y = \text{l}.\sqrt[4]{y},$

$$\frac{1}{n}\,\text{l}.y = \text{l}.\sqrt[n]{y};$$

from which it follows, that *the logarithm of any root of a number is the logarithm of the number itself divided by the index of the root.*

Reduction of Exponential Equations by Logarithms.

(113.) If we take the general equation $b^x = c$, we shall have, by supposing the logarithms of both members known,

$$\text{l}.b^x = \text{l}.c.$$

It has been shown, that the logarithm of any power of a number, is equal to the logarithm of the number itself, multiplied by the exponent of the power (111); hence, $\text{l}.b^x = x\,\text{l}.b$, and therefore we have, $x\,\text{l}.b = \text{l}.c$;

dividing by $\text{l}.b$, $x = \frac{\text{l}.c}{\text{l}.b}$.

For a numerical example; let it be required to find the value of x from the equation $8^x = 128$, we shall have,

$$x = \frac{\text{l}.128}{\text{l}.8} = \frac{2{,}10721}{0{,}90309} = 2{,}3333, \text{ \&c.}$$

(114.) Having shown how numerical calculations are performed by means of logarithms, we will now consider, more particularly, the properties of the common system of logarithms, formed upon the base 10.

We have seen, that in this system, the log. of $1 = 0$, $\text{l}.10 = 1$, $\text{l}.100 = 2$, $\text{l}.1000 = 3$, $\text{l}.10000 = 4$, and that all intermediate numbers have for their logarithms, the sam whole numbers, plus a fraction. The integral part of a lo rithm can, therefore, be known from the number of plac figures contained in the number to which it bel numbers less than 10, it is 0; for those greater than 10, and less than 100, it is 1; for those greater than 100, and less than 1000, it is 2, &c., being always one less than the number of figures expressing the given number. The integral part of a logarithm is called the *characteristic* of the logarithm.

(115.) It may also be shown, that the decimal part of the logarithms of numbers, one of which is 10, 100, 1000, &c. times greater or less than the other, is the same; for the logarithms of 10, 100, 1000, &c. are whole numbers, and being added or subtracted, would only alter the value of the characteristic, and would not affect the decimal part.

For example; 1.28×10, or $1.280 = 1.28 + 1.10$,
$= 1{,}447158 + 1. = 2{,}447158$,
1.28×100, or $1.2800 = 1{,}447158 + 2. = 3{,}447158$,
1.28×1000, or $1.28000 = 1{,}447158 + 3. = 4{,}447158$.

For the same reason, the logarithm of any number of figures, whether whole numbers or decimals, will be the same, with the exception of the characteristic, which will always be one less than the number of figures in the integral part of the number. For example; taking the number 654685, and dividing it successively by 10, and for each division subtracting from its logarithm the logarithm of 10, we have,

Number.	Log.
654685	5,8150324
65468,5	4,8150324
6546,85	3,8150324
654,685	2,8150324
65,4685	1,8150324
6,54685	0,8150324
0,654685	— 1,8150324
0,0654685	— 2,8150324

From this example, it also appears, that the *characteristic* of a decimal fraction is negative. It is always determined by the place which the first significant figure of the decimal occupies, counting from the decimal point towards the right. If it occupy the first place, the characteristic is —1; if the second, it is —2, &c., being obtained by subtracting the logarithm of 10, 100, 1000, &c. from zero.

(116.) The logarithms of all fractional numbers are negative, since they result from the subtraction of a greater logarithm from a less. To obtain the logarithm of $\frac{2}{3}$, for example, we have (110), l. 2 — l. 3 = l. $\frac{2}{3}$,

or, 0,3010300 — 0,4771213 = — 0,1760913.

Negative logarithms would be inconvenient in calculation, and to avoid their use we reduce the fraction to a decimal, the logarithm of which is positive, except the characteristic.

To effect this reduction, we have only to consider the numerator multiplied by 10, 100, or 1000, as may be necessary, which is done by adding 1, 2, or 3 to its characteristic, and then subtract the logarithm of the denominator. The number of units necessary to be added to the characteristic of the numerator, will determine the characteristic of the resulting logarithm, which must always, as has been shown (115), be negative.

Thus, to obtain the logarithm of $\frac{2}{3}$, we multiply the numerator by 10, which gives for the numerator, 20 tenth parts of a unit, which divided by 3, will give a quotient in tenth parts. To effect this division by logarithms, we subtract the log. of 3 from the log. of 20 :—

l. 20 = 1,3010300
l. 3 = 0,4771213
0,8239087

and since this difference is the log. of 10th parts, we must affix to it — 1 for a characteristic ; or, which is the same thing, considering this difference the log. of a whole number, subtract the log. of 10 from it, to effect the division by 10, which gives — 1 for the characteristic.

In practice, it is usual to perform the subtraction, as though the logarithm of the numerator were the greater, till we come to the characteristic, we then take the excess of the characteristic of the log. of the denominator over that of the numerator, and affect it with the negative sign, for the characteristic of the resulting logarithm. The result is the same as by the preceding method.

For example ; let it be required to find the quotient of 5, divided by 73 in decimals, by means of logarithms.

The log. of 5 is	0,6989700
and the log. of 73	1,8633229
	— 2,8356471 difference.

The number corresponding to this logar

nearly; but the characteristic, -2, shows that this number must be a decimal fraction, the first significant figure of which stands in the place of hundredths (115); we have, therefore, 0,068491 for the quotient of 5 divided by 73, extended to six places of decimals.

(117.) We now proceed to show the application of logarithms to algebraic formulas. The student should be provided with a table of logarithms, and exercise himself in their use by giving numerical values to the algebraic quantities, and performing the operations indicated by means of logarithms.

Let it be required to find the last term of a geometrical progression by means of logarithms; the first term, the ratio, and the number of terms being given. Calling a the first term, r the ratio, and n the number of terms, we have (94), $l=ar^{n-1}$; taking the logarithms $\mathrm{l}.l=\mathrm{l}.a+(n-1)\times\mathrm{l}.r$ (109), (121).

If l, r, and n be known, we find, $a=\frac{l}{r^{n-1}}$. And taking the logarithms $\mathrm{l}.a=\mathrm{l}.l-(n-1)\mathrm{l}.r$ (110).

If a, l, and n be given, and we would find r, we have,

$$r=\sqrt[n-1]{\frac{l}{a}} \quad . \quad . \quad (94);$$

and by logarithms, $\mathrm{l}.r=\frac{\mathrm{l}.l-\mathrm{l}.a}{n-1}$ . . (112).

If a, l, and r be known, and we would find n, we have,

$$r^{n-1}=\frac{l}{a};$$

and by logarithms, $(n-1)\mathrm{l}.r=\mathrm{l}.l-\mathrm{l}.a$ . . (109);

whence, $n-1=\frac{\mathrm{l}.l-\mathrm{l}.a}{\mathrm{l}.r}$ . . (113).

We might also take the formula $S=\frac{a(r^n-1)}{r-1}$;

and supposing three of the quantities which enter into it known.

Find the other by means of logarithms. Thus, n, a, and r being known, we have, $S=\frac{a(r^n-1)}{r-1}$.

And by logarithms $\mathrm{l}.S=\mathrm{l}.a+\mathrm{l}.(r^n-1)-\mathrm{l}.(r-1)$.

Supposing S, a, and r, known, $r^n=\frac{rS-S+a}{a}$;

and by logarithms, $n\,\mathrm{l}.r = \mathrm{l}.(rS - S + a) - \mathrm{l}.a$,

$$\text{and,}\quad n = \frac{\mathrm{l}.(rS - S + a) - \mathrm{l}.a}{\mathrm{l}.r} \quad . \; . \quad (113).$$

Proportion by Logarithms.

(118.) Let there be the proportion $a : b = c : d$; then we shall have $d = \frac{bc}{a}$, and by logarithms, $\mathrm{l}.d = \mathrm{l}.b + \mathrm{l}.c - \mathrm{l}.a$; hence, to find the fourth term of a proportion, *add together the logarithms of the 2d and 3d terms, and subtract from their sum the logarithm of the first term; the remainder will be the logarithm of the fourth term.*

FORMULAS RELATING TO INTEREST AND ANNUITIES.

Simple Interest.

(119.) Let the principal, or sum on which interest is to be calculated $= a$, the rate per cent., or the fraction expressing the interest of $1 for one year $= r$; the time for which the principal a is loaned $= t$; and the amount at the expiration of the time $t = A$; then we shall have the proportion as $1 is to r, its interest for one year, so is the principal a, to its interest for one year, or, $1 : r = a : ar$; ar is therefore the interest of a for one year. But if the principal is loaned for a time greater or less than one year, the interest will be proportional to the time which will give the proportion, $1 : ar = t : art$; art would therefore be the interest of a for the time t. Thus, if $a = 230$, $r = ,06$, and $t = 2$ years and 5 months, we shall have, $art = 230 \times ,06 \times 2\frac{5}{12} = 33,35$. If, the other values remaining the same, $t = 67$ days, then $art = 230 \times ,06 \times \frac{67}{365} = 2,533$.

(120.) If we would know the amount, we have only to add the principal to the interest, and we shall have $A = a + art$. From this last formula, others may be derived by bringing each of the quantities which enter into it, to stand alone on one side of the sign of equality; thus,

$$a = \frac{A}{1 + rt};\; r = \frac{A - a}{at},\ \text{and}\ t = \frac{A - a}{ar}.$$

From the first of these formulas, we can find the principal, the amount, the rate per cent., and the time being given; from the second we can find the rate per cent., the amount, the principal, and the time being given; and from the third, we can find the time, the rate, the principal, and the amount being given. Thus, if we would know what principal would amount to $425, in 7 months, at 6 per cent., we should have by substituting these values of A, r, and t, in the first formula,

$$a = \frac{425}{1 + ,06 \times \frac{7}{12}} = \frac{425}{1,035} = 410,628.$$

If it were required to find the rate per cent. at which $648 would amount to $727 38, in 2 years and 4 months, we should have, from the second formula,

$$r = \frac{727,38 - 648}{648 \times 2\frac{1}{3}} = \frac{79,38}{1512} = ,0525, \text{ or } 5\tfrac{1}{4} \text{ per cent.}$$

For an application of the third formula, let the time be required in which $116 would amount to $130 at 5 per cent.; and we shall have, $t = \frac{130 - 116}{116 \times 05} = \frac{14}{5,8} = 2\frac{12}{29}$ years, or 2 years and 5 months, nearly.

Compound Interest.

(121.) In compound interest, the interest is supposed to remain in the hands of the borrower, and being added to the principal, to form a part of the principal for the succeeding year. Calling a' the amount for the first year, we shall have the proportion, as $1 is to its amount, $1 + r$, so is a, the given sum, to its amount for one year; or $1 : 1 + r = a : a'$. a' therefore equals $a + ar$, or $a(1 + r)$, which is the principal for the second year. Calling a'' the amount of this new principal for one year, we have $1 : 1 + r = a(1 + r) : a''$; whence $a'' = a(1 + r)^2$.

In like manner calling a''' the amount of this last principal for a year, we shall have, $a''' = a(1 + r)^3$; and since the amount for each year is found by multiplying that of the preceding year by $1 + r$, it is evident that the amount for any number of years will be found by multiplying the principal by that power of $1 + r$, which is denoted by the number of years. Calling A the amount for n years, we shall therefore have the general formula

$$A = a(1 + r)^n,$$

From this formula, by reducing the equation with reference to a, r, and n, respectively, we derive the three following:—

$$a=\frac{A}{(1+r)^n};\quad 1+r=\sqrt[n]{\frac{A}{a}},\quad n=\frac{l.A-l.a}{l.(1+r)} \qquad (113)$$

The application of the four preceding formulas to numerical examples, is most conveniently made by logarithms. By taking the logarithms of the quantities which enter into them, they become, respectively,

$$l.A=l.a+n\,l.(1+r); \quad . \quad . \quad (109),\ (111)$$

$$l.a=l.A-n\,l.(1+r); \quad . \quad . \quad (110),\ (111)$$

$$l.(1+r)=\frac{l.A-l.a}{n}; \quad . \quad . \quad (110),\ (112)$$

$$n=\frac{l.A-l.a}{l.(1+r)}. \quad . \quad . \quad (110),\ (113)$$

Let it be required to find the amount of $1250, for 20 years at 6 per cent. By the first formula we have,

$A=1250(1+,06)^{20}$, or, $l.A=l.1250+20\times l.(1+,06)$.[*]

Taking the logarithms, we have,

$$l.A = 3,0969100+0,5061180 = 3,6030280;$$

From which we find, $A = \$4008,92$.

For another example; find the amount of $1000 for 3 years at 5 per cent.

$$A=1000(1+,05)^3=1000(1,05)^3=1000\times 1,157625$$
$$= 1157,625.$$

For an example of the second formula; let it be required to find the principal, which, in 25 years, at 7 per cent., compound interest, will amount to $5000.

By logarithms, $l.a = l.\,5000-25\times l.(1,07)$,

or, $l.a = 3,698970-25\times 0,029384 = 2,964370$;

from which we find, $a = \$921,234$.

If we would know at what rate per cent. $1000 would amount to $3361,85 in 20 years, we have from the third formula, $l.(1+r)=\dfrac{l.3361,85-l.1000}{20}=\dfrac{3,526578-3,}{20}$

$$= 0,0263289,$$

from which we find,

$$1+r = 1,0625, \text{ and } r = ,0625.$$

$6\frac{1}{4}$ per cent. is therefore the rate of interest.

Lastly, if it were required to find the time in which \$500 would amount to \$1039,46, at 5 per cent., we have, by the fourth formula,

$$n = \frac{\mathrm{l}.1039{,}46 - \mathrm{l}.500}{\mathrm{l}.1{,}05} = \frac{3{,}0168095 - 2{,}6989700}{0{,}0211893}.$$

$$= \frac{0{,}3178395}{0{,}0211893} = 15.$$

Examples for Practice.

1. What is the amount of \$1200 at compound interest, for 10 years, at 6 per cent. Ans. \$2149,19.

2. What principal will amount to \$4000, in 30 years, at 5 per cent. Ans. \$925,51.

3. At what rate per cent. will \$1000 amount to \$1479,15 in 6 years. Ans. 6¾.

4. In how may years will money, at compound interest, at 6 per cent, amount to double, treble, and quadruple, the original sum.

Ans. 11,8955 years, 18,8145 years, and 23,791 years, respectively.

ANNUITIES.

(122.) An annuity is a sum of money payable periodically, for some specified time, or during the life of the receiver. If the payments are not made, the annuity is said to be in *arrear*, and the receiver is entitled to interest on the several payments in arrear.

The *worth* of an annuity in arrear, is the sum of the several payments, together with compound interest on every payment after it became due. The *present worth* of an annuity is the sum which, at compound interest, would amount to a sum equal to the worth of the annuity, for the time specified, supposing it to be in arrear. We now proceed to investigate formulas to be applied to calculations relating to annuities.

(123.) Let $a =$ the annuity, $A =$ the amount of the annuity in arrear, $r =$ the rate per cent, and $n =$ the number of years that the annuity has been forborn. It is evident that on the last payment due, no interest would accrue. It will therefore be a. The one preceding will be the amount of a

for one year, $a(1+r)$; the one before this, the amount of a, for two years, $a(1+r)^2$; and so on to the first, which will be the amount of a for $n-1$ years, $a(1+r)^{n-1}$. The several sums due may therefore be truly represented by a progression, the first term of which is a, the ratio $(1+r)$, and the number of terms n; the worth of the annuity will be the sum of this progression, and we shall have,

$$A = a + a(1+r) + a(1+r)^2 + a(1+r)^3 \dots + a(1+r)^{n-1};$$

or, taking the sum of the progression (97),

$$A = \frac{a[(1+r)^n - 1]}{(1+r)-1} = \frac{a[(1+r)^n - 1]}{r}.$$

This formula furnishes the means of finding the worth of an annuity in arrear. Let it be required to find the worth of an annuity of \$50 per annum, that has been forborn 6 years, at an interest of 6 per cent., then we shall have,

$$A = \frac{50[(1+06)^6 - 1]}{,06}, \text{ or } \mathrm{l}.A = \mathrm{l}.50 + \mathrm{l}.[(1+,06)^6 - 1] - \mathrm{l}.06.$$

The value of $(1,06)^6$, as found by logarithms, is 1,41852, subtracting 1 from this, as indicated in the formula, and taking the logarithm of the remainder, we have,

$$\begin{aligned} \mathrm{l}.A &= 1,6989700 + (-1,6217162) - (-2,7781513) \\ &= 2,542534; \end{aligned}$$

from which we find, $A = 348,767$.

(124.) Since the present worth of an annuity is that sum, which at compound interest, would amount, at the termination of the annuity, to the worth of the whole annuity in arrear, we shall have, by calling the present worth A'.

$$A'(1+r)^n = \frac{a[(1+r)^n - 1]}{r}.$$

From this formula we can deduce three others, severally expressing the values of A', a, and n, in terms of the others,

$$A' = \frac{a[(1+r)^n - 1]}{r(1+r)^n};\quad a = \frac{A'r(1+r)^n}{(1+r)^n - 1},$$

$$n\,\mathrm{l}.(1+r) = \mathrm{l}.a - \mathrm{l}.(a - A'r), \text{ or } n = \frac{\mathrm{l}.a - \mathrm{l}.(a - A'r)}{\mathrm{l}.(1+r)}.$$

Let it be required to find the present worth of an annuity of $200 per annum, to continue 12 years at 6 per cent. By the first formula we have,

$$A' = \frac{200[(1,06)^{12} - 1]}{,06(1,06)^{12}};$$

or, $\text{l.}a = \text{l.}200 + \text{l.}[(1,06)^{12} - 1] - \text{l.},06 - 12 \times \text{l.}1,06$; from which we may find the value of A' to be $1675,76.

For an application of the second formula, find the annuity that would cancel a debt of $1000 in 12 years, with interest at 5 per cent.,

$$a = \frac{1000 \times ,05(1,05)^{12}}{(1,05)^{12} - 1};$$

or, $\text{l.}a = \text{l.}1000 + \text{l.},05 + 12 \times \text{l.}1,05 - \text{l.}[(105)^{12} - 1]$. By performing the operations indicated, we find $a = 112,825$. Lastly; let it be required to find the time that would be necesssary to pay a debt of $210,707, in annual payments of $30 each, with interest, at 7 per cent. By the third formula we have,

$$n = \frac{\text{l.}30 - \text{l.}(30 - 210,707 \times ,07)}{\text{l.}1,07} = \frac{0,2938369}{0,0293838} = 10, \text{ nearly.}$$

If the annuity falls due oftener than once a year, we have only to alter the value of r, so as to make it the rate per cent. for the time between two payments. Thus, if the annuity is due quarterly, and the interest is 6 per cent., $r = ,015$; if it be due semi-annually, $r = ,03$. Suppose it required to find what sum must be paid per month to cancel a debt of $500, in 5 years at 6 per cent., we have, $r = \frac{,06}{12} = ,005$; and $n = 60$, the number of months in 5 years. Then

$$a = \frac{500 \times ,005(1,005)^{60}}{(1,005)^{60} - 1} = 9,66628.$$